JOURNAL OF APPROXIMATION THEORY 38, 105–138 (1983)

Perfect Splines of Minimum Norm for Monotone Norms and Norms Induced by Inner Products, with Applications to Tensor Product Approximations and n-Widths of Integral Operators

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Communicated by Oved Shisha

Received May 22, 1981; revised October 29, 1982

Existence and a partial characterization of perfect splines of minimum norm, related to totally positive kernels, are obtained by a unified method applicable to the class of monotone norms (norms for which $|f(x)| \leqslant |g(x)|$ implies $\|f\| \leqslant \|g\|$). The method is based on the duality between this problem and best L^1-approximation, which provides a pointwise improvement theorem for perfect splines. Similar results are obtained for norms induced by inner products by the equivalence between this case and the self-dual case of perfect splines of minimum L^1-norm. The knots and zeros of the minimal perfect splines are then used in choosing best tensor-product approximations to totally positive kernels in a norm which is a tensor product of a monotone norm and the L^1-norm. Also n-widths and optimal spaces in the sense of Kolmogorov and Gelfand are obtained for integral operators with totally positive kernels via the minimal perfect splines. These results generalize known results for the L^p-norms, $1 \leqslant p \leqslant \infty$, to monotone norms and to norms induced by inner products.

1. INTRODUCTION

Perfect splines of minimum norm are of central importance in the theory of n-width for classes of functions related to integral operators with totally positive (TP) kernels, and also in optimal tensor product approximations to such kernels.

Given a TP kernel $K(x, y) \in C([a, b] \times [c, d])$, a perfect spline with knots $c = y_0 < y_1 < \cdots < y_n < y_{n+1} = d$ is defined as

$$\phi_{\mathbf{y}}(x) = \int_c^d K(x, y)\, h_{\mathbf{y}}(y)\, dy, \qquad (-1)^i\, h_{\mathbf{y}}(y) > 0,$$
$$y_i < y < y_{i+1}, \quad 0 \leqslant i \leqslant n, \tag{1.1}$$

with $h_{\mathbf{y}} \in L^\infty[c, d]$.

0021-9045/83 $3.00

Existence of perfect splines of minimum L^p-norms, $1 \leqslant p < \infty$, leading to various n-width results, is proved in [12] for TP kernels satisfying certain additional assumptions on the independence of their sections. Also optimal tensor product approximations to the kernel K in norms of the form

$$|||f(x,y)||| = \left[\int \left[\int_c^d |f(x,y)|\, dy \right]^p dx \right]^{1/p} \tag{1.2}$$

are obtained in [12]. The case $p = \infty$ is investigated in [11].

The present work provides a unified method for the derivation of the existence and a characterization of perfect splines of minimum norm. The method we use is independent of the explicit form of the norm and applies to the wide class of monotone norms (norms for which $|f| \leqslant |g|$ on $[a,b]$ implies $\|f\| \leqslant \|g\|$). Consequently also n-widths results and tensor product optimal approximations to bivariate functions are obtained for monotone norms.

Our method of proof is based on the duality between the problem of perfect splines of minimum norm and best L^1 approximation. This duality is the key to an "improvement theorem" for perfect splines, in analogy to known "improvement theorems" for monosplines [6, 15]. The knots of the perfect spline of minimum norm are characterized as a fixed point in R^n of an "improving" transformation, based on the canonical points for best L^1 approximation by weak Chebyshev systems [10]. This transformation is used in [2] to prove the uniqueness of perfect splines of minimum L^1-norm for a certain class of kernels.

For extended TP kernels this approach yields the existence and a characterization of perfect splines of minimum norm for all monotone norms. For TP kernels such results are obtained only for various subclasses of monotone norms, depending on the smoothness and the independence of the sections $K(\cdot, y)$, $K(x, \cdot)$ of the kernels. In particular it is shown that for a certain subclass of monotone norms containing the L^p-norms, $1 \leqslant p < \infty$, the present method of proof requires weaker assumptions on the kernels than those needed for the proof in [12].

The self-dual case of perfect splines of minimum L^1-norm related to symmetric kernels, with its specific structure, provides a tool in the analysis of perfect splines of minimum norm for norms induced by inner products. The latter case is proved to be equivalent to the former under certain conditions on the kernel and the inner product.

The results on perfect splines of minimum norm and the ideas in the proofs are analogous to those in [15] dealing with monosplines of minimum norm corresponding to extended TP kernels. The central role of best one-sided L^1-approximation in the analysis of the monospline case is replaced by best two-sided L^1-approximation in the present analysis.

In Section 2 we introduce the main concepts and notations, cite several results, and derive the correspondence between the properties of the kernel and the set of monotone norms for which the forthcoming analysis applies. In Section 3 we obtain the existence and a characterization of the minimal perfect splines for monotone norms, and also the specific structure of the self-dual case corresponding to the L^1-norm. Using this structure we investigate in Section 4 perfect splines of minimum norm for norms induced by inner products.

The results of Section 3 are used in Section 5 in the derivation of best tensor product approximations $\sum_{i=1}^{n} u_i(x)\, v_i(y)$ to TP kernels $K(x, y)$ in norms of the form

$$|||f||| \equiv \left\| \int_c^d |f(\cdot, y)|\, h(y)\, dy \right\|,$$

where $\|\cdot\|$ is a monotone norm, and $h \geqslant 0$, $h \in L^\infty[c, d]$.

Also in this section n-widths of Kolmogorov and Gelfand type are computed via the results of Section 3 and 4 for two classes of functions,

$$K_h = \left\{ \int_c^d K(x, y)\, \sigma(y)\, dy \mid \sigma \in L^\infty[c, d], |\sigma(y)| \leqslant h(y), y \in [c, d] \right\},$$

$$K_X = \left\{ \int_c^d K(x, y)\, g(y)\, dy \mid g \in X, \|g\| \leqslant 1 \right\},$$

where $X \subset L^1[c, d]$ is normed by a monotone norm $\|\cdot\|$.

For the class K_h under the assumptions of Section 3, we obtain the two types of n-widths, with respect to monotone norms by a method similar to that in [12]. For the Kolmogorov n-width the extra assumption of strict convexity of the monotone norm is required. Without this assumption on the monotone norm, the n-width of Kolmogorov type is obtained only for the restricted set of approximiting functions $\{K(\cdot, y) \mid y \in (c, d)\}$. This is done by the characterization of perfect splines of minimum norm as minimal functions from the wider class of functions

$$\left\{ \phi_y + \sum_{i=1}^{n} a_i K(x, y_i) \mid a_i \in R, i = 1,..., n, c < y_1 < \cdots < y_n < d \right\}.$$

Both types of n-widths are found to equal the norm of the minimal perfect splines and the corresponding optimal spaces are related either to the set of knots or the set of zeros of a minimal perfect spline. Similar n-width results for the class K_h are obtained, under the assumptions of Section 4, for inner product norms. Results on the n-width of the image of the unit ball in inner product spaces are well known (see, e.g., [9] and references therein). The results here for the class K_h are new and extend those known for the L^2-norm [12].

For the class K_X the n-widths are computed with respect to the L^1-norm by introducing a dual norm to that defining K_X. The results in this case are dual to those in the previous case: the roles of the Kolmogorov and Gelfand n-width are interchanged, and the n-widths and optimal spaces are related to perfect splines corresponding to the kernel K^{T}, which are minimal with respect to the dual norm.

2. Notations and Preliminary Results

In this section we introduce several concepts and notations, cite three results, and derive several preliminary results necessary for the proof of the existence and characterization of perfect splines of minimum norm.

A central concept to this work is that of totally positive kernels [5].

Definition 2.1. A kernel $K(x, y) \in C([a, b] \times [c, d])$ is termed *totally positive* (TP) if for any $\mathbf{x} = (a \leqslant x_1 \cdots < x_n \leqslant b)$, $\mathbf{y} = (c \leqslant y_1 < \cdots < y_n \leqslant d)$, $n \geqslant 1$,

$$\begin{vmatrix} K(x_1, y_1) & \cdots & K(x_1, y_n) \\ \vdots & & \vdots \\ K(x_n, y_1) & \cdots & K(x_n, y_n) \end{vmatrix} \geqslant 0. \tag{2.1}$$

If in (2.1) there is strict inequality for all choices of $\mathbf{x}$ and $\mathbf{y}$ as above, the kernel is termed strictly totally positive (STP).

Any n functions of the form $\{K(x, y_i), i = 1,..., n\}$ or $\{K(x_i, y), i = 1,..., n\}$ constitute a weak Chebyshev system if K is TP and a Chebyshev system if K is STP [5].

Definition 2.2. A system of functions $U = \{u_1,..., u_n\}$ is a weak Chebyshev system on $[a, b]$ if $U \subset C[a, b]$ and

$$\begin{pmatrix} u_1,..., u_n \\ x_1,..., x_n \end{pmatrix} \equiv \begin{vmatrix} u_1(x_1) & \cdots & u_n(x_1) \\ \vdots & & \vdots \\ u_1(x_n) & \cdots & u_n(x_n) \end{vmatrix} \geqslant 0 \tag{2.2}$$

for any $a \leqslant x_1 < \cdots < x_n \leqslant b$. The system U is termed a Chebyshev system if there is always strict inequality in (2.2). The dimension of U is the dimension of the span of the functions in U.

For $U = \{u_1,..., u_n\}$ a weak Chebyshev system, $C^{\pm}(U)$ denotes the cone of functions $\{f\}$ such that $\{u_1,..., u_n, \varepsilon f\}$ is also a weak Chebyshev system for $\varepsilon = \pm 1$, $C(U) = C^+(U) \cup C^-(U)$.

For $K \in C^{r,0}([a, b] \times [c, d])$ and for $\mathbf{x} = (a \leqslant x_1 \leqslant \cdots \leqslant x_m \leqslant b)$ with no more than $r+1$ repetitions of the same point and for $\mathbf{y} = (c < y_1 < \cdots < y_n < d)$, we introduce the following notations:

(i) The matrix

$$K \begin{pmatrix} x_1, \ldots, x_m \\ y_1, \ldots, y_n \end{pmatrix} \equiv \left(\frac{\partial^{l_i}}{\partial x^{l_i}} K(x_i, y_j) \right)_{i=1,\ j=1}^{m \quad n} \tag{2.3}$$

with

$$l_i = \max\{\nu \mid x_{i-\nu} = x_i\}. \tag{2.4}$$

A kernel K is termed ETP of degree $r+1$ in x on $[a, b] \times (c, d)$ if determinants (2.3) are positive.

(ii) The two systems of functions

$$K(\mathbf{y}) = \{K(x, y_i), i = 1, \ldots, n\}, \tag{2.5}$$

$$K[\mathbf{x}] = \left\{ \frac{\partial^{l_i}}{\partial x^{l_i}} K(x_i, y), i = 1, \ldots, m \right\}, \tag{2.6}$$

which are weak Chebyshev systems in case K is TP [5].

Following [12] we introduce the concept of nondegeneracy of a kernel K. Here we need a refinement of this concept with regard to multiplicities of points.

Definition 2.3. A set of points $a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b$ is termed to be of *order* r if points in (a, b) are repeated at most r times and the endpoints $\{a, b\}$ at most once.

Definition 2.4. A kernel $K \in C^{r,s}([a, b] \times [c, d])$ is nondegenerate of order r in x on (a, b) (on $[a, b]$) if for any $n \geqslant 1$ and any $\mathbf{x} = (a < x_1 \leqslant \cdots \leqslant x_n < b)$, $(\mathbf{x} = (a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b))$ of order $r+1$, the dimension of $K[\mathbf{x}]$ is n.

A similar definition holds for nondegeneracy in y. For the case of nondegeneracy of order 0 in both variables on $(a, b) \times (c, d)$, this property is weaker than the one in [12].

The technique used in the next section for the investigation of perfect splines of minimum norm is independent of the particular nature of the norm and is applicable to monotone norms defined by the property

$$f, g \in C[a, b], \qquad |f(x)| \leqslant |g(x)|, \qquad x \in [a, b] \Rightarrow \|f\| \leqslant \|g\|.$$

Norms for which $\|f\| < \|g\|$ with f, g as above and $f \not\equiv g$ are termed "strictly monotone." The L^p-norms $1 \leqslant p < \infty$ are strictly monotone.

Best approximation by monotone norms are studied in [7, 8], where the following two results are proved:

RESULT A [7]. *Let* $\|\cdot\|$ *be a monotone norm defined on* $C[a,b]$ *and let* $f, g \in C[a,b]$ *satisfy* $|f(x)| \leqslant |g(x)|$, $x[a,b]$, *with equality only at the zeros of* g. *Then* $\|f\| < \|g\|$ *whenever* $g \not\equiv 0$.

RESULT B [8]. *Let* $\{u_1,..., u_n\} \subset C[a,b]$ *be such that* $\{u_1,..., u_n\}$ *and* $\{u_1,..., u_{n-1}\}$ *are Chebyshev systems on* $[a,b]$ *and let* u^* *be a best approximation to* $f \in C[a,b]$ *from the span of* $\{u_1,..., u_n\}$ *in a monotone norm. If* $f-u^*$ *has only isolated zeros in* $[a,b]$, *then*

$$\tilde{Z}_{[a,b]}(f-u^*) \geqslant n,$$

where $\tilde{Z}_{[a,b]}(f)$ *denotes the number of isolated zeros of* f *in* $[a,b]$, *counting twice zeros in* (a,b) *where* f *does not change sign.*

This count of zeros in case of general monotone norms requires in the forthcoming study of perfect splines of minimum norm a finer analysis than that needed for the L^p-norms, $1 \leqslant p \leqslant \infty$, for which $\tilde{Z}$ in the above result can be replaced by the number of simple zeros in (a,b).

For ease of formulation we refer in the following to a system of functions and to its span by the same symbol. The following result is concerned with best approximation in the L^1-norm by weak Chebyshev systems:

RESULT C [10]. *Let* $U = \{u_1,..., u_n\} \subset C[a,b]$ *be a weak Chebyshev system of dimension* n *on* $[a,b]$ *such that for every* $a < x_1 < \cdots < x_n < b$

$$\operatorname{span}\{f(x_1),..., f(x_n) \mid f \in C(U)\} = R^n.$$

Then for any $g(x) \geqslant 0$, $\operatorname{meas}\{x \mid g(x) = 0\} = 0$, there exists a unique set of points $a = \tau_0 < \tau_1 < \cdots < \tau_n < \tau_{n+1} = b$ with the property

$$\sum_{i=0}^{n} (-1)^i \int_{\tau_i}^{\tau_{i+1}} u_j(x)\, g(x)\, dx = 0, \qquad j = 1,..., n. \tag{2.7}$$

For this set of points

$$\det \begin{pmatrix} u_1,..., u_n \\ \tau_1,..., \tau_n \end{pmatrix} > 0 \tag{2.8}$$

and the best approximation to any $f \in C(U)$ from U in the L^1_g-norm

$$\|f-u^*\|_{1,g} \equiv \int_a^b |f-u^*|\, g = \inf_{u \in U} \int_a^b |f-u|\, g$$

is the unique interpolant to f *from* U *at the points* $\tau_1,..., \tau_n$.

Remark 2.1. For $K(x,y)$ TP on $[a,b]\times[c,d]$ and nondegenerate of order 0 in y on (c,d), and for any $a\leqslant x_1\leqslant\cdots\leqslant x_n\leqslant b$ such that $K[\mathbf{x}]$ is a weak Chebyshev system of dimension n, the system $K[\mathbf{x}]$ satisfies the assumptions of Result C. To see this, observe that $K(x,\cdot)\in C(K[\mathbf{x}])$ for $x\in(a,b)$, and that for any $c<y_1<\cdots<y_n<d$

$$\operatorname{span}\{(K(x,y_1),\ldots,K(x,y_n))\mid x\in(a,b)\}=R^n$$

by the nondegeneracy of K in y on (c,d).

In Lemmas 2.1 and 2.2 we assume that $K\in C^{r,0}([a,b]\times[c,d])$ is a TP kernel, nondegenerate of order r in x on (a,b) and of order 0 in y on (c,d).

LEMMA 2.1. *Let*

$$\phi_{\mathbf{y}}(x)=\int_c^d K(x,y)f(y)\,dy,$$

where $f\in L^\infty[c,d]$ *satisfies* $(-1)^i f(y)\geqslant 0$, $y_i<y<y_{i+1}$, $i=0,\ldots,n$, *for* $c=y_0<y_1<\cdots<y_n<y_{n+1}=d$. *Then* $\phi_{\mathbf{y}}\in C(K(y))$. *Moreover if* $\operatorname{meas}\{y\mid f(y)=0\}=0$, *then any function of the form*

$$u(x;\mathbf{y},\mathbf{a})\equiv\phi_{\mathbf{y}}(x)+\sum_{i=1}^n a_iK(x,y_i) \tag{2.9}$$

has at most n zeros in (a,b) *counting multiplicities up to order* $r+1$.

This lemma can be proved by the method used for the case $r=0$ in [11, Lemma 7.1]. As a direct conclusion we obtain

LEMMA 2.2. *Under all the assumptions of Lemma* 2.1, *the set of zeros* $a<x_1\leqslant x_2\leqslant\cdots\leqslant x_l<b$, *with multiplicities counted up to order* $r+1$, *of any function of form* (2.9), *has the property that* $l\leqslant n$ *and*

$$\operatorname{rank} K\begin{pmatrix}x_1,\ldots,x_l\\ y_1,\ldots,y_n\end{pmatrix}=l. \tag{2.10}$$

Moreover, if $l=n$, *then for any* $x\in(a,b)-\{x_1,\ldots,x_n\}$

$$\operatorname{sgn}[u(x;\mathbf{y},\mathbf{a})]=(-1)^{i(x)},\qquad \textit{where}\quad i(x)=\max\{i\mid x_i<x\},\quad x_0=a. \tag{2.11}$$

Proof. Assume to the contrary that for a function v of form (2.9) with zeros $a<x_1\leqslant x_1\leqslant\cdots\leqslant x_l<b$ there exists $m<l$ such that

$$\operatorname{rank}\ K\begin{pmatrix}x_{i_1},\ldots,x_{i_m}\\ y_1,\ldots,y_n\end{pmatrix}=m=\operatorname{rank} K\begin{pmatrix}x_1,\ldots,x_l\\ y_1,\ldots,y_n\end{pmatrix}, \tag{2.12}$$

and choose $a < \xi_1 < \cdots < \xi_{n-m} < b$ to satisfy $\{\xi_1,..., \xi_{n-m}\} \subset (a, b) - \{x_1,..., x_l\}$, and

$$\det K \begin{pmatrix} x_{i_1},..., x_{i_m}, \xi_1,..., \xi_{n-m} \\ y_1,..., y_n \end{pmatrix} \neq 0. \tag{2.13}$$

This choice is possible by (2.12) and the assumptions on the nondegeneracy of K.

In view of (2.13) there exists $u \in K(\mathbf{y})$ interpolating v at $x_{i_1},..., x_{i_m}$, $\xi_1,..., \xi_{n-m}$. This u vanishes at $x_{i_1},..., x_{i_m}$ and by (2.12) also at $x_1,..., x_l$, and therefore u interpolates v at $x_1,..., x_l$, $\xi_1,..., \xi_{n-m}$. Thus $v - u$, which is also of form (2.9), has $n - m + l > n$ zeros, counting multiplicities up to order $r + 1$, in contradiction to Lemma 2.1.

Now if $l = n$, then by (2.10)

$$\Delta = \det K \begin{pmatrix} x_1,..., x_n \\ y_1,..., y_n \end{pmatrix} \neq 0,$$

and

$$u(x; \mathbf{y}, \mathbf{a}) = \frac{1}{\Delta} \det \begin{pmatrix} \phi_{\mathbf{y}}, K(\cdot, y_1),..., K(\cdot, y_n) \\ x, x_1,..., x_n \end{pmatrix}$$
$$= \frac{1}{\Delta} \int_c^d \det K \begin{pmatrix} x, x_1,..., x_n \\ y, y_1,..., y_n \end{pmatrix} f(y)\, dy,$$

proving (2.11). ∎

Remark 2.2. Under the stronger assumption that K is also nondegenerate of order r in x on the closed interval $[a, b]$, and/or of order 0 in y on $[c, d]$, Lemmas 2.1, 2.2. hold for zeros $a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b$ of order $r + 1$ in $[a, b]$, and/or for $c \leqslant y_1 < \cdots < y_n \leqslant d$.

A direct consequence of Result B, Lemma 2.2, and Remark 2.2 is

COROLLARY 2.1. *Let* $K \in C^{1,0}([a, b] \times [c, d])$ *be STP and nondegenerate of order* 1 *in* x *on* $[a, b]$, *and let* $\phi_{\mathbf{y}}$ *be defined as in Lemma* 2.1. *Then any best approximation* u^* *to* $\phi_{\mathbf{y}}$ *from* $K(\mathbf{y})$ *in any monotone norm satisfies*

$$\tilde{Z}_{[a,b]}(\phi_{\mathbf{y}} - u^*) = n. \tag{2.14}$$

Moreover, the zeros $a \leqslant \xi_1 \leqslant \cdots \leqslant \xi_n \leqslant b$ *counted in* (2.14) *satisfy*

$$\det K \begin{pmatrix} \xi_1,..., \xi_n \\ y_1,..., y_n \end{pmatrix} \neq 0.$$

This property of the best approximation to the perfect spline $\phi_{\mathbf{y}}$, expressed in the last corollary, is essential in the analysis of perfect splines of minimum norm. Since such a property cannot be stated for all TP kernels and all monotone norms, in the absence of an analogous result to Result B for weak Chebyshev systems, we associate an appropriate class of norms to each TP kernel.

DEFINITION 2.5. For a given TP kernel $K(x, y) \in C^{r,0}([a, b] \times [c, d])$, the class of norms $N(K)$ consists of all the monotone norms with the property: For any $c < y_1 < \cdots < y_n < d$ such that $K(\mathbf{y})$ is of dimension n, there exists a best approximation v^* to $\phi_{\mathbf{y}}$ from $K(\mathbf{y})$ such that $\phi_{\mathbf{y}} - v^*$ has n zeroes $a \leqslant \xi_1 \leqslant \cdots \leqslant \xi_n \leqslant b$ of order $r+1$ satisfying

$$\det K \begin{pmatrix} \xi_1, \dots, \xi_n \\ y_1, \dots, y_n \end{pmatrix} \neq 0. \tag{2.15}$$

Thus the class $N(K)$ consists of all the monotone norms whenever K is STP on $[a, b] \times [c, d]$, and nondegenerate of order 1 in x on $[a, b]$. Of course this is also the case when K is extended totally positive.

Furthermore, this definition and Lemma 2.2. imply

COROLLARY 2.2. *Let $K \in C([a, b] \times [c, d])$ be TP and nondegenerate of order* 0 *in x and y on (a, b) and (c, d) respectively. Then $N(K)$ contains all the monotone norms of L^p type, namely, the monotone norms with the property that for any $f \in C[a, b]$ and any weak Chebyshev system of dimension n, $U \subset C[a, b]$, there exists a best approximation to f from U, interpolating f at n distinct points in (a, b). In particular, $N(K)$ contains all the L^p-norms, $1 \leqslant p \leqslant \infty$.*

It is possible to prove that for TP kernels in $C^{1,0}([a, b] \times [c, d])$ which are nondegenerate of order 1 in x on $[a, b]$ and of order 0 in y on (c, d), the class $N(K)$ contains the wide class of decomposable monotone norms. These are norms with the property that for any $m > 0$ disjoint open intervals in (a, b), $I_1, \dots, I_m$, it is possible to find points, $a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b$, $n \geqslant m+1$, containing all the boundary points of $I_1, \dots, I_m$, such that

$$\|f\|_{[a,b]} = N_{n,\mathbf{x}}\{\|f\|_{[x_0, x_1]}, \dots, \|f\|_{[x_n, x_{n+1}]}\}. \tag{2.16}$$

In (2.16) $N_{n,\mathbf{x}}$ is a monotone norm defined on R^{n+1} and $\|f\|_{[x_i, x_{i+1}]}$ is a monotone norm of the restriction of f to $[x_i, x_{i+1}]$.

An example of decomposable monotone norms is furnished by norms of the following type:

$$\|f\| = \left[\sum_{i=0}^{s} (\|f\|_{p_i})^q\right]^{1/q}, \qquad \|f\|_{p_i} = \|f\|_{L^{p_i}[\xi_i,\xi_{i+1}]}, \tag{2.17}$$

where $1 \leqslant p_0,..., p_s, q \leqslant \infty$, and $a = \xi_0 < \xi_1 < \cdots < \xi_s < \xi_{s+1} = b$ are fixed.

For decomposable monotone norms a weaker version of Result B holds in the context of best approximation by weak Chebyshev systems. This will be shown elsewhere.

In the following section we derive the results for monotone norms in the class $N(K)$ by a unified method independent of the particular form of the norm. For the L^p-norms $1 \leqslant p < \infty$, this method of proof applies to the class of TP kernels satisfying the assumptions of Corollary 2.2 which is wider than the class considered in [12].

3. Existence and Characterization of Perfect Splines of Minimum Monotone Norm

Let $K \in C^{r,0}([a, b] \times [c, d])$, $r \geqslant 0$, be a TP kernel nondegenerate of order 0 in y on (c, d) and of order r in x on (a, b) or $[a, b]$, and let $h \in L^\infty[c, d]$ satisfy

$$h \geqslant 0 \quad \text{on} \quad [c, d], \qquad \text{meas}\{y \mid h(y) = 0\} = 0. \tag{3.1}$$

For any $\mathbf{y}$ in the n simplex

$$S^n \equiv S^n[c, d] \equiv \{\boldsymbol{\eta} \mid c = \eta_0 < \eta_1 < \cdots < \eta_n < \eta_{n+1} = d\} \tag{3.2}$$

we define the function

$$h_{\mathbf{y}}(y) = (-1)^j h(y), \qquad y_j < y < y_{j+1}, \quad j = 0,..., n, \tag{3.3}$$

and the corresponding perfect spline

$$\phi_{\mathbf{y}} = \int_c^d K(\cdot, y)\, h_{\mathbf{y}}(y)\, dy. \tag{3.4}$$

The function

$$F(\mathbf{y}) \equiv F(y_1,..., y_n) = \|\phi_{\mathbf{y}}\|, \tag{3.5}$$

for $\|\cdot\|$ a monotone norm, is a continuous function of $\mathbf{y}$ in S^n, by the continuity of the norm $\|\cdot\|$ and by the continuous dependence of $\phi_{\mathbf{y}}$ on $\mathbf{y}$ in S^n. The function $F(\mathbf{y})$ can be extended continuously to the closure of S^n

$$\overline{S^n} = \{\boldsymbol{\eta} \mid c \leqslant \eta_1 \leqslant \cdots \leqslant \eta_n \leqslant d\} \tag{3.6}$$

according to the following definition:

$$\|\phi_{\mathbf{y}}\| = F(y_1,..., y_n) \equiv \left\| \int_c^d K(\cdot, y)\, h_{\mathbf{z}}(y)\, dy \right\| = \|\phi_{\mathbf{z}}\|, \tag{3.7}$$

where $\mathbf{z} \in S^k$, $k \leqslant n$, and $z_1,..., z_k$ are those points among $y_1,..., y_n$ which are interior to $[c, d]$ and appear with odd multiplicities. Hence there exists $\mathbf{y}^* \in S^m$, $m \leqslant n$, satisfying

$$\|\phi_{\mathbf{y}^*}\| \leqslant \|\phi_{\mathbf{y}}\| \qquad \text{for all} \quad \mathbf{y} \in S^k, \quad k \leqslant n. \tag{3.8}$$

In the following we restrict the discussion to monotone norms in the class $N(K)$, and show that a point $\mathbf{y}^* \in S^m$, $m \leqslant n$, satisfied (3.8) only if $\mathbf{y}^* \in S^n$. A new characterization of such $\mathbf{y}^*$ is also given.

LEMMA 3.1. *Let $c \leqslant z_1 < \cdots < z_k \leqslant d$, and let $a \leqslant \xi_1 \leqslant \cdots \leqslant \xi_k \leqslant b$ be of order $r+1$ and satisfy*

$$\Delta \equiv \det K \begin{pmatrix} \xi_1,..., \xi_k \\ z_1,..., z_k \end{pmatrix} > 0. \tag{3.9}$$

Denote by $v \in K(\mathbf{z})$ the unique interpolant to $\phi_{\mathbf{z}}$ at $\tilde{\boldsymbol{\xi}} = (\xi_1,..., \xi_k)$

$$v^{(l_i)}(\xi_i) = \phi_{\mathbf{z}}^{(l_i)}(\xi_i), \qquad l_i = \max\{v \mid \xi_{i-v} = \xi_i\}, \quad i = 1,..., k.$$

Then if $k < n$ or if $k = n$ and $v \neq 0$, there exists a perfect spline $\phi_{\mathbf{y}}$, $\mathbf{y} \in S^n$, satisfying

$$|\phi_{\mathbf{y}}(x)| \leqslant |\phi_{\mathbf{z}}(x) - v(x)|, \qquad x \in [a, b], \qquad \phi_{\mathbf{y}} \neq \phi_{\mathbf{z}}. \tag{3.10}$$

Moreover in the following two cases:

(i) *K is* ETP *of degree $r+1$ in x on $[a, b]$,*

(ii) *K is* STP *on $(a, b) \times (c, d)$ and $a < \xi_1 < \cdots < \xi_n < b$,* (3.10) *holds with equality only at the zeros of $\phi_{\mathbf{z}} - v$.*

Proof. If $k < n$ choose $\xi_{k+1},..., \xi_n$ such that $\boldsymbol{\xi} = (\xi_1,..., \xi_n) \in \overline{S^n}[a, b]$ and dimension $K[\boldsymbol{\xi}] = n$. This is possible by the nondegeneracy of K in x on (a, b). In case $k = n$, obviously $\boldsymbol{\xi} = \tilde{\boldsymbol{\xi}}$. By (3.9)

$$\begin{aligned} |\phi_{\mathbf{z}}(x) - v(x)| &= \frac{1}{\Delta} \det \begin{pmatrix} \phi_{\mathbf{z}}, K(\cdot, z_1),..., K(\cdot, z_k) \\ x, \xi_1,..., \xi_k \end{pmatrix} \\ &= \frac{1}{\Delta} \left| \int_c^d \det K \begin{pmatrix} y, z_1,..., z_k \\ x, \xi_1,..., \xi_k \end{pmatrix} h_{\mathbf{z}}(y)\, dy \right| \\ &= \frac{1}{\Delta} \int_c^d \left| \det K \begin{pmatrix} y, z_1,..., z_k \\ x, \xi_1,..., \xi_k \end{pmatrix} \right| h(y)\, dy \\ &= \|K(x, y) - u_x(y)\|_{1,h} \geqslant 0, \end{aligned} \tag{3.11}$$

where u_x is the unique function in $K[\tilde{\xi}]$ interpolating $K(x,\cdot)$ at $\mathbf{z}=(z_1,..., z_k)$. Hence there is equality in (3.11) if and only if $K(x,\cdot)\in K[\tilde{\xi}]$. Since $K[\xi]$ is a weak Chebyshev system of dimension n, Result C guarantees the existence of $\mathbf{y}\in S^n$ such that

$$\int_c^d u(y)\,h_{\mathbf{y}}(y)\,dy=0, \qquad u\in K[\xi], \tag{3.12}$$

or equivalently the perfect spline $\phi_{\mathbf{y}}$ satisfies

$$\phi_{\mathbf{y}}^{(l_i)}(\xi_i)=\int_c^d \frac{\partial^{l_i}}{\partial x^{l_i}} K(\xi_i,y)\,h_{\mathbf{y}}(y)\,dy=0, \qquad i=1,..., n. \tag{3.13}$$

Obviously in case $k<n$, $\mathbf{y}\neq\mathbf{z}$. If $k=n$ and $v\not\equiv 0$, the uniqueness of the function $v\in K(\mathbf{z})$ interpolating $\phi_{\mathbf{z}}$ at $\xi_1,..., \xi_n$ implies that $\mathbf{y}\neq\mathbf{z}$, as $u\equiv 0$ interpolates $\phi_{\mathbf{y}}$ at ξ.

Furthermore, Result C implies that for any $x\in[a,b]$ the function $u_x^*\in K[\xi]$, interpolating $K(x,\cdot)\in C(K[\xi])$ at $y_1,..., y_n$, satisfies

$$\|K(x,\cdot)-u_x^*\|_{1,h}\leqslant\|K(x,\cdot)-u_x\|_{1,h} \tag{3.14}$$

with equality if and only if $K(x,\cdot)\in K[\tilde{\xi}]$, namely at the zeros of $\phi_{\mathbf{z}}-v$. But by the definition of u_x^* and by (3.12)

$$\|K(x,\cdot)-u_x^*\|_{1,h}=\left|\int_c^d [K(x,y)-u_x^*(y)]\,h_{\mathbf{y}}(y)\,dy\right|=|\phi_{\mathbf{y}}(x)|, \tag{3.15}$$

which in view of (3.14) and (3.11) completes the proof of (3.10).

In the special cases (i), (ii) $u_x^*\neq u_x$ whenever $K(x,\cdot)-u_x\not\equiv 0$, since this function can have no more than its k zeros at $\mathbf{z}$, while $K(x,\cdot)-u_x^*$ vanishes at $\mathbf{y}\neq\mathbf{z}$. Hence there is equality in (3.14) and therefore in (3.10) if and only if $K(x,\cdot)\in K[\tilde{\xi}]$, namely at the zeros of $\phi_{\mathbf{z}}-v$. ∎

THEOREM 3.1a (An improvement theorem for perfect splines). *Let $c<z_1<\cdots<z_k<d$, $k\leqslant n$, be such that either $k<n$, or $k=n$ and $v\equiv 0$ is not a best approximation to $\phi_{\mathbf{z}}$ from $K(\mathbf{z})$. Then there exists n points $c<y_1<\cdots<y_n<d$ with the property*

$$\|\phi_{\mathbf{y}}\|<\|\phi_{\mathbf{z}}\| \tag{3.16}$$

whenever K is TP *and nondegenerate of order* 1 *in x on $[a,b]$ and of order* 0 *in y on $(c,d]$ (or $[c,d)$) and the norm $\|\cdot\|$ is in $N(K)$.*

Proof. In case zero is not a best approximation to $\phi_{\mathbf{z}}$ from $K(\mathbf{z})$, let $v \not\equiv 0$ be such a best approximation. Then

$$\|\phi_{\mathbf{z}} - v\| < \|\phi_{\mathbf{z}}\| \tag{3.17}$$

and v satisfies the assumptions of Lemma 3.1 since $\|\cdot\| \in N(K)$. The case that zero is a best approximation to $\phi_{\mathbf{z}}$ from $K(\mathbf{z})$ occurs only for $k < n$ by the assumptions of the theorem. In this case let v denote a best approximation to $\phi_{\mathbf{z}}$ from the subspace $K(\mathbf{z}) \cup K(\cdot, d)$ (or $K(\mathbf{z}) \cup K(\cdot, c)$) which is of dimension $k+1 \leqslant n$. Necessarily $v \not\equiv 0$, since $\|\cdot\| \in N(K)$ and therefore $\phi_{\mathbf{z}} - v$ has $k+1$ zeros of order $r+1$ in $[a, b]$, while by Remark 2.2, $\phi_{\mathbf{z}}$ has at most k zeros of order $r+1$ in $[a, b]$. Hence (3.17) holds for this v as well, and v satisfies the conditions of Lemma 3.1. Hence Lemma 3.1 guarantees the existence of points $c < y_1 < \cdots < y_n < d$ such that

$$|\phi_{\mathbf{y}}(x)| \leqslant |\phi_{\mathbf{z}}(x) - v(x)|, \qquad x \in [a, b].$$

By the monotonicity of the norm we finally obtain

$$\|\phi_{\mathbf{y}}\| \leqslant \|\phi_{\mathbf{z}} - v\| < \|\phi_{\mathbf{z}}\|. \quad \blacksquare \tag{3.18}$$

The requirements on the nondegeneracy of K, involving the boundaries of the intervals, can be relaxed, if either the kernel or the norm have more structure.

THEOREM 3.1b. *The improvement result of Theorem* 3.1a *is also valid in the following cases*:

(i) K *is* TP *and nondegenerate of order* 0 *in* x *on* (a, b) *and in* y *on* $(c, d]$ *or* $[c, d)$, *and the norm* $\|\cdot\|$ *is of* L^p *type.*

(ii) K *is* TP *and nondegenerate of order* 0 *in* x *on* (a, b) *and in* y *on* (c, d), *and the norm* $\|\cdot\|$ *is strictly monotone of* L^p *type.*

(iii) K *is* TP *on* $[a, b] \times (c, d)$ *and the norm* $\|\cdot\| \in N(K)$ *is strictly monotone.*

(iv) K *is* STP *on* $(a, b) \times (c, d)$ *and the norm* $\|\cdot\|$ *is of* L^p *type.*

(v) K *is* ETP *of degree* 2 *in* x *on* $[a, b] \times (c, d)$ *and the norm* $\|\cdot\|$ *is monotone.*

Proof. The proof of case (i) is the same as the proof of Theorem 3.1a. For all the rest of the cases the improvement result as expressed by (3.16) is due to a strict inequality in the first rather than the second inequality in (3.18). Let v be a best approximation to $\phi_{\mathbf{z}}$ from $K(\mathbf{z})$. Then v satisfies the requirements of Lemma 3.1: in case (v) by Result B in cases (iii), (iv) by the structure of the norm and in cases (ii) by the structure of the norm and by Lemma 2.2. For cases (iv), (v) Lemma 3.1 guarantees that

$$|\phi_{\mathbf{y}}(x)| \leqslant |\phi_{\mathbf{z}}(x) - v(x)|, \qquad x \in [a, b]$$

with equality only at the zeros of $\phi_{\mathbf{z}} - v$. Hence by Result A

$$\|\phi_{\mathbf{y}}\| < \|\phi_{\mathbf{z}} - v\| \leqslant \|\phi_{\mathbf{z}}\|.$$

For cases (ii), (iii) the strict inequality above follows from (3.10) and the strict monotonicity of the norm. ∎

As a direct consequence of Theorem 3.1 and its proof we obtain

THEOREM 3.2. *Let $\phi_{\mathbf{y}^*}$, $\mathbf{y}^* \in S^m$, $m \leqslant n$, be a perfect spline of minimum norm*

$$\|\phi_{\mathbf{y}^*}\| \leqslant \|\phi_{\mathbf{y}}\|, \qquad \mathbf{y} \in S^k, \quad k \leqslant n. \tag{3.19}$$

Then under the assumptions of Theorem 3.1 $m = n$,

$$c < y_1^* < \cdots < y_n^* < d, \tag{3.20}$$

and

$$\|\phi_{\mathbf{y}^*}\| \leqslant \left\| \phi_{\mathbf{y}^*} - \sum_{i=1}^{n} a_i K(\cdot, y_i^*) \right\|. \tag{3.21}$$

Moreover $\phi_{\mathbf{y}^*}$ has property (3.19) *if and only if*

$$\|\phi_{\mathbf{y}^*}\| \leqslant \left\| \phi_{\mathbf{y}} - \sum_{i=1}^{k} a_i K(x, y_i) \right\|, \qquad \mathbf{y} \in S^k, \quad k \leqslant n. \tag{3.22}$$

Remark 3.1. An improvement theorem, as Theorem 3.1, can be obtained for *positive* perfect splines if v^* in (3.17) is replaced by a best approximation $\hat{v} \in K(\mathbf{z})$ to $\phi_{\mathbf{z}}$ from below. This result holds for monotone norms in the class $N^+(K)$ defined as $N(K)$ in Definition 2.5 but with regard to best approximation from below. Consequently, an analogous result to Theorem 3.2 for positive perfect splines holds, namely, any positive perfect spline of minimum norm $\phi_{\mathbf{y}^*}$ corresponds to $\mathbf{y}^* \in S^n$ and satisfies

$$\|\phi_{\mathbf{y}^*}\| \leqslant \|\phi_{\mathbf{y}} - u\|, \qquad u \in K(\mathbf{y}), \quad \phi_{\mathbf{y}} - u \geqslant 0, \quad \mathbf{y} \in S^k, \quad k \leqslant n,$$

with equality only if $u \equiv 0$ and $\mathbf{y} \in S^n$.

Property (3.21) of perfect splines of minimum norm yields in the case of the sup-norm the existence of exactly $n+1$ points of alternation of ϕ_{y^*}, $a \leqslant x_0 < \cdots < x_n \leqslant b$:

$$\phi_{y^*}(x_i) = (-1)^i \|\phi_{y^*}\|_\infty, \qquad i = 0,..., n, \tag{3.23}$$

since $u = 0$ is the best approximation to ϕ_{y^*} from $K(\mathbf{y}^*)$. Property (3.23) of ϕ_{y^*} implies that $u = 0$ is also a best approximation in the sup-norm to ϕ_{y^*} from $K(\mathbf{y})$ for all $\mathbf{y} \in S^m$, $m \leqslant n$ (see also [13]).

The results of Theorems 3.1 and 3.2 for the L^p-norms, $1 \leqslant p < \infty$, apply in view of Corollary 2.2 to all TP kernels which are nondegenerate of order 0 in x and y on (a, b) and (c, d), respectively. The rest of this section is concerned with such kernels and with the specific structure of perfect splines of minimum L_g^1-norm, with g satisfying

$$g \in L^\infty[a, b], \qquad g(x) \geqslant 0, \qquad x \in [a, b], \qquad \operatorname{meas}\{x \mid g(x) = 0\} = 0. \tag{3.24}$$

THEOREM 3.3. *Let ϕ_{y^*} for $\mathbf{y}^* \in S^n$ satisfy*

$$\|\phi_{y^*}\|_{1,g} \leqslant \|\phi_{\mathbf{y}}\|_{1,g} \qquad \textit{for any} \quad \mathbf{y} \in S^k[c, d], \quad k \leqslant n. \tag{3.25}$$

Then there exist n points $a = x_0^* < x_1^* < \cdots < x_{n+1}^* = b$ such that

$$\operatorname{sgn}[\phi_{y^*}(x)] = \operatorname{sgn}[g_{x^*}(x)], \qquad x \in [a, b] - \{x_1^*,..., x_n^*\}, \tag{3.26}$$

where

$$g_{x^*}(x) = (-1)^j g(x), \qquad x_j^* < x < x_{j+1}^*, \quad j = 0,..., n. \tag{3.27}$$

Moreover, the perfect spline

$$\psi_{x^*}(y) = \int_a^b K(x, y)\, g_{x^*}(x)\, dx \tag{3.28}$$

satisfies

$$\operatorname{sgn}[\psi_{x^*}(y)] = \operatorname{sgn}[h_{y^*}(y)], \qquad y \in [c, d] - \{y_1^*,..., y_n^*\}, \tag{3.29}$$

$$\|\psi_{x^*}\|_{1,h} = \|\phi_{y^*}\|_{1,g} \leqslant \|\psi_{\mathbf{x}}\|_{1,h}, \qquad \mathbf{x} \in S^k[a, b], \quad k \leqslant n. \tag{3.30}$$

Proof. By Theorem 3.2, zero is the best approximation to ϕ_{y^*} from $K(\mathbf{y}^*)$ in the L_g^1-norm. In view of Result C, there exist n points $\mathbf{x}^* = (x_1^*,..., x_n^*) \in S^n[a, b]$ with the property

$$\int_a^b K(x, y_i^*)\, g_{x^*}(x)\, dx = 0, \qquad i = 1,..., n, \tag{3.31}$$

and $\phi_{\mathbf{y}^*}$ vanishes at these points, each being a simple zero. This proves (3.26). In terms of the perfect spline (3.28), relations (3.31) are

$$\psi_{\mathbf{x}^*}(y_i^*) = 0, \qquad i = 1,..., n.$$

This together with Lemma 2.2 implies (3.29), and therefore

$$\|\psi_{\mathbf{x}^*}\|_{1,h} = \int_c^d \psi_{\mathbf{x}^*}(y)\, h_{\mathbf{y}^*}(y)\, dy = \int_c^d \int_a^b K(x, y)\, g_{\mathbf{x}^*}(x)\; h_{\mathbf{y}^*}(y)\, dx\, dy. \tag{3.32}$$

Similarly, by (3.26)

$$\|\phi_{\mathbf{y}^*}\|_{1,g} = \int_a^b \phi_{\mathbf{y}^*}(x)\, g_{\mathbf{x}^*}(x)\, dx = \int_a^b \int_c^d K(x, y)\, g_{\mathbf{x}^*}(y)\, h_{\mathbf{y}^*}(x)\, dy\, dx. \tag{3.33}$$

Thus $\|\psi_{\mathbf{x}^*}\|_{1,h} = \|\phi_{\mathbf{y}^*}\|_{1,g}$. To complete the proof of (3.30) we use the symmetry between the two problems:

$$\min_{\substack{\mathbf{y} \in S^m[c,d] \\ m \leqslant n}} \|\phi_{\mathbf{y}}\|_{1,g}; \qquad \min_{\substack{\mathbf{x} \in S^m[a,b] \\ m \leqslant n}} \|\psi_{\mathbf{x}}\|_{1,h}.$$

By the first part of (3.30) applied to the second problem,

$$\|\psi_{\boldsymbol{\xi}^*}\|_{1,h} \equiv \min_{\substack{\mathbf{x} \in S^m[a,b] \\ m \leqslant n}} \|\psi_{\mathbf{x}}\|_{1,h} = \|\phi_{\boldsymbol{\eta}^*}\|_{1,g}, \qquad \boldsymbol{\eta}^* \in S^n[c, d],$$

and therefore

$$\|\phi_{\mathbf{y}^*}\|_{1,g} = \|\psi_{\mathbf{x}^*}\|_{1,h} \geqslant \|\psi_{\boldsymbol{\xi}^*}\|_{1,h} = \|\phi_{\boldsymbol{\eta}^*}\|_{1,g} \geqslant \|\phi_{\mathbf{y}^*}\|_{1,g},$$

implying the second part of (3.30). ∎

In [2] it is proved that (3.21) for the L_g^1-norm is a necessary and sufficient condition for (3.25) and that the point $\mathbf{y}^* \in S^n[c, d]$ is unique for special choices of K, g, h.

The case $K(x, y) = K(y, x)$, $[c, d] = [a, b]$, and $g(x) \equiv h(x)$ has a specific self-dual structure:

THEOREM 3.4. *Let $K(x, y) = K(y, x)$ on $[a, b]^2$ and let*

$$h \in L^\infty[a, b], \qquad h \geqslant 0, \quad \text{meas}\{x \mid h(x) = 0\} = 0. \tag{3.34}$$

Then for any perfect spline $\phi_{\mathbf{y}^*} = \int_a^b K(\cdot, y)\, h_{\mathbf{y}^*}(y)\, dy$ of minimum L_h^1-norm the set of knots $\mathbf{y}^*$ coincides with the set of zeros:

$$\text{sgn}\left[\int_a^b K(x, y)\, h_{\mathbf{y}^*}(y)\, dy\right] = \text{sgn}[h_{\mathbf{y}^*}(x)], \qquad x \in [a, b] - \{y_1^*,..., y_n^*\}. \tag{3.35}$$

Proof. By Theorem 3.3 and the symmetry of $K(x, y)$, there exists $\mathbf{x}^* \in S^n[a, b]$ such that

$$\operatorname{sgn}\left[\int_a^b K(x,y)\, h_{\mathbf{y}^*}(y)\, dy\right] = \operatorname{sgn}[h_{\mathbf{x}^*}(x)], \qquad x \in [a,b] - \{x_1^*,..., x_n^*\}, \tag{3.36}$$

$$\operatorname{sgn}\left[\int_a^b K(x,y)\, h_{\mathbf{x}^*}(y)\, dy\right] = \operatorname{sgn}[h_{\mathbf{y}^*}(x)], \qquad x \in [a,b] - \{y_1^*,..., y_n^*\}. \tag{3.37}$$

We shall prove that $\mathbf{x}^* = \mathbf{y}^*$. Assume to the contrary that $\mathbf{x}^* \neq \mathbf{y}^*$, and introduce the bilinear form

$$[f, e] = \int_a^b \int_a^b K(x,y) f(x)\, e(y)\, dx\, dy. \tag{3.38}$$

This bilinear form is a semi-inner product since $K(x, y)$ is symmetric and TP, and therefore the Schwarz inequality holds,

$$[f, e]^2 \leqslant [f, f\,][e, e]. \tag{3.39}$$

On the other hand, by (3.36), (3.37), and the assumption $\mathbf{x}^* \neq \mathbf{y}_*$,

$$\int_a^b \left[\int_a^b K(x,y)\, h_{\mathbf{y}^*}(y)\, dy\right] h_{\mathbf{x}^*}(x)\, dx > \int_a^b \left[\int_a^b K(x,y)\, h_{\mathbf{y}^*}(y)\, dy\right] h_{\mathbf{y}^*}(x)\, dx,$$

$$\int_a^b \left[\int_a^b K(x,y)\, h_{\mathbf{x}^*}(y)\, dy\right] h_{\mathbf{y}^*}(x)\, dx > \int_a^b \left[\int_a^b K(x,y)\, h_{\mathbf{x}^*}(y)\, dy\right] h_{\mathbf{x}^*}(x)\, dx,$$

which in terms of the semi-inner product (3.38) become

$$[h_{\mathbf{y}^*}, h_{\mathbf{x}^*}] > [h_{\mathbf{y}^*}, h_{\mathbf{y}^*}], \qquad [h_{\mathbf{x}^*}, h_{\mathbf{y}^*}] > [h_{\mathbf{x}^*}, h_{\mathbf{x}^*}],$$

in contradiction to (3.39). Therefore $\mathbf{x}^* = \mathbf{y}^*$ and the proof of the theorem is completed. ∎

The result in Theorem 3.4 is of central importance in the derivation of existence and characterization of perfect splines of minimum norm for norms induced by inner products.

Remark 3.2. The results of this section can be extended to perfect splines of the more general form

$$\int_c^d K(x,y)\, h_{\mathbf{y}}(y)\, dy + \sum_{j=0}^{r+1} \alpha_j K_j(x, c), \qquad \boldsymbol{\alpha} \in R^{r+2}, \tag{3.40}$$

with $K_j(x,y)=(\partial^j/\partial y^j)K(x,y)$, under the assumptions that $K_{r+1}(\cdot,c)\in C[a,b]$ and $\{K_j(x,c),\ j=0,\dots,r+1\}\cup K(\mathbf{y})$ is of dimension $n+r+2$ for any $\mathbf{y}\in S^n$. This from with $K(x,y)=(x-y)_+^{r+1}$, $h\equiv 1$, $c=0$, corresponds to the classical algebraic perfect splines

$$\sum_{i=0}^{r+1} a_i x^i + \frac{x^{r+2}}{(r+2)} + 2\sum_{i=1}^{n} (-1)^i \frac{(x-x_i)_+^{r+2}}{(r+2)}. \tag{3.41}$$

The minimal perfect spline of form (3.40) can be characterized by an improvement theorem as Theorem 3.1. This requires two simple modifications in the proof of Theorem 3.1:

(a) The weak Chebyshev system $K(\mathbf{z})$ is replaced by the weak Chebyshev system

$$K(\mathbf{c}\cup\mathbf{z})\equiv\{K(x,z_i),\ i=1,\dots,k,\ K_j(x,c),\ j=0,\dots,r+1\}. \tag{3.42}$$

(b) If $\tilde{\boldsymbol{\xi}}\in\overline{S^{k+r+2}}[a,b]$ is the vector of zeroes of $\phi_{\mathbf{z}}-v^*$, then in Lemma 3.1 $\mathbf{y}\in S^n[c,d]$ are the canonical points of Result C for the weak Chebyshev space of dimension n, consisting of those functions in $K[\boldsymbol{\xi}]$, $\boldsymbol{\xi}\in S^{n+r+2}[a,b]$ ($\tilde{\boldsymbol{\xi}}\subset\boldsymbol{\xi}$), which have a zero of multiplicity $r+2$ in c.

4. Perfect Splines of Minimum Norm for Norms Induced by Inner Products

For a given inner product (,), real or complex, defined on functions with domain D_1, we consider perfect splines of minimum norm for the norm

$$\|f\|^2=(f,f). \tag{4.1}$$

These perfect splines are related to perfect splines of minimum L_h^1-norm if the following conditions hold:

(i) $K(z,y)$ is defined on a domain $D_1\times D_2$, where D_2 contains a real interval $[a,b]$.

(ii) Any perfect spline

$$\phi_{\mathbf{y}}(z)=\int_a^b K(z,y)\,h_{\mathbf{y}}(y)\,dy, \qquad \mathbf{y}\in S^k,\quad k\leqslant n, \tag{4.2}$$

with h satisfying (3.34), has a finite norm.

(iii) The kernel

$$G(x,y)=(K(\cdot,x),K(\cdot,y)) \tag{4.3}$$

is real, continuous, totally positive on $[a, b]^2$, and nondegenerate of order 0 in x and in y on $(a, b)^2$.

Under these assumptions

$$G(x, y) = G(y, x), \qquad x, y \in [a, b], \tag{4.4}$$

$$(\phi_{\mathbf{y}}, K(\cdot, x)) = \int_a^b G(x, y)\, h_{\mathbf{y}}(y)\, dy \equiv \varphi_{\mathbf{y}}(x), \tag{4.5}$$

$$(\phi_{\mathbf{y}}, \phi_{\mathbf{x}}) = \int_a^b \varphi_{\mathbf{y}}(x)\, h_{\mathbf{x}}(x)\, dx. \tag{4.6}$$

An example of kernels K and inner products (,) satisfying assumptions (i)–(iii) is furnished by the complex-valued kernels $(1 - z\bar{\zeta})^{-1}$, $(1 - z\bar{\zeta})^{-2}$ defined on $|z| < 1$, $|\zeta| < 1$, and the corresponding inner products [14]:

$$(f, g) = \int_{|z|=1} f(z)\, \bar{g}(z)\, ds, \qquad (f, g) = \int_{|z|<1} f(z)\, \overline{g(z)}\, dx\, dy.$$

Lemma 4.1. *Let* $u(z) = \sum_{i=1}^k a_i K(z, y_i)$, $\mathbf{y} \in S^k[a, b]$, $k \leqslant n$, *be a best approximation to* $\phi_{\mathbf{y}}$ *in the norm* (4.1) *from* $K(\mathbf{y})$. *Then*

$$\left\| \phi_{\mathbf{y}} - \sum_{i=1}^k a_i K(\cdot, y_i) \right\|^2 = \left\| \varphi_{\mathbf{y}}(x) - \sum_{i=1}^k a_i G(x, y_i) \right\|_{1,h}. \tag{4.7}$$

Proof. As a best approximation from a finite-dimensional subspace in an inner-product norm, u satisfies the normal equations

$$\left(\phi_{\mathbf{y}} - \sum_{i=1}^k a_i K(\cdot, y_i), K(\cdot, y_j) \right) = 0, \qquad j = 1, \ldots, k, \tag{4.8}$$

which in view of (4.3)–(4.5) become

$$\varphi_{\mathbf{y}}(y_j) - \sum_{i=1}^k a_i G(y_j, y_i) = 0, \qquad j = 1, \ldots, k. \tag{4.9}$$

Hence by Lemma 2.2

$$\operatorname{sgn} \left[\varphi_{\mathbf{y}}(x) - \sum_{i=1}^k a_i G(x, y_i) \right] = \operatorname{sgn}[h_{\mathbf{y}}(x)], \qquad x \in [a, b] - \{y_1, \ldots, y_k\}. \tag{4.10}$$

Combining (4.8) with (4.10) and recalling (4.5) and (4.6), we obtain

$$\left\|\phi_{\mathbf{y}}-\sum_{i=1}^{k} a_i K(\cdot, y_i)\right\|^2 = \left(\phi_{\mathbf{y}}-\sum_{i=1}^{k} a_i K(\cdot, y_i), \phi_{\mathbf{y}}\right)$$

$$= \int_a^b \left[\varphi_{\mathbf{y}}(x) - \sum_{i=1}^{k} a_i G(x, y_i)\right] h_{\mathbf{y}}(x)\,dx = \left\|\varphi_{\mathbf{y}}-\sum_{i=1}^{k} a_i G(\cdot, y_i)\right\|_{1,h}. \quad \blacksquare$$

In the following, $S^n \equiv S^n[a,b]$. As an immediate consequence of Lemma 4.1 we obtain

LEMMA 4.2. *Let* $\varphi_{\mathbf{y}^*}, \mathbf{y}^* \in S^n$, *satisfy*

$$\|\varphi_{\mathbf{y}^*}\|_{1,h} \leqslant \|\varphi_{\mathbf{y}}\|_{1,h}, \qquad \mathbf{y} \in S^k, \quad k \leqslant n. \tag{4.11}$$

Then

$$\|\phi_{\mathbf{y}}\|^2 \geqslant \|\varphi_{\mathbf{y}^*}\|_{1,h} = \|\phi_{\mathbf{y}^*}\|^2, \qquad \mathbf{y} \in S^k, \quad k \leqslant n. \tag{4.12}$$

Proof. By Lemma 4.1 and Theorem 3.2, for $\mathbf{y} \in S^k$, $k \leqslant n$,

$$\|\phi_{\mathbf{y}}\|^2 \geqslant \|\phi_{\mathbf{y}} - u\|^2 = \|\varphi_{\mathbf{y}} - v\|_{1,h} \geqslant \|\varphi_{\mathbf{y}^*}\|_{1,h},$$

where u is the best approximation to $\phi_{\mathbf{y}}$ from $K(\mathbf{y})$, and $v = (u(z), K(z,\cdot)) \in G(\mathbf{y})$. To complete the proof we show that $\|\varphi_{\mathbf{y}^*}\|_{1,h} = \|\phi_{\mathbf{y}^*}\|^2$. By Theorem 3.4 applied to the kernel $G(x,y)$

$$\operatorname{sgn}[\varphi_{\mathbf{y}^*}(x)] = \operatorname{sgn}[h_{\mathbf{y}^*}(x)], \qquad x \in [a,b] - \{y_1^*, ..., y_n^*\},$$

and therefore in view of (4.6)

$$\|\varphi_{\mathbf{y}^*}\|_{1,h} = \int_a^b \varphi_{\mathbf{y}^*}(x)\, h_{\mathbf{y}^*}(x)\,dx = (\phi_{\mathbf{y}^*}, \phi_{\mathbf{y}^*}) = \|\phi_{\mathbf{y}^*}\|^2. \quad \blacksquare$$

Lemma 4.2 implies the existence of $\phi_{\mathbf{y}^*}$ with $\mathbf{y}^* \in S^n$ such that

$$\|\phi_{\mathbf{y}^*}\| \leqslant \|\phi_{\mathbf{y}}\|, \qquad \mathbf{y} \in S^k, \quad k \leqslant n. \tag{4.13}$$

The proof of the equivalence between the two problems

$$\min_{\substack{\mathbf{y}\in S^k \\ k\leqslant n}} \|\phi_{\mathbf{y}}\|; \qquad \operatorname*{mim}_{\substack{\mathbf{y}\in S^k \\ k\leqslant n}} \|\varphi_{\mathbf{y}}\|_{1,h} \tag{4.14}$$

is completed in

LEMMA 4.3. *Let* $\phi_{\mathbf{y}^*}$ *with* $\mathbf{y}^* \in S^m$, $m \leqslant n$, *satisfy* (4.13). *Then* $m = n$ *and* $\varphi_{\mathbf{y}^*}$ *satsifies* (4.11).

Proof. Let u be the best approximation to $\phi_{\mathbf{y}^*}$ from $K(\mathbf{y}^*)$, and let $v = (u(z), K(z, \cdot))$. Then by Lemmas 4.1 and 4.2

$$\|\varphi_{\mathbf{y}^*}\|^2 \geqslant \|\phi_{\mathbf{y}^*} - u\|^2 = \|\varphi_{\mathbf{y}^*} - v\|_{1,h} \geqslant \|\varphi_{\boldsymbol{\eta}^*}\|_{1,h} = \|\phi_{\boldsymbol{\eta}^*}\|^2, \qquad (4.15)$$

where $\varphi_{\boldsymbol{\eta}^*}$ is a perfect spline of minimum L_h^1-norm. Recalling definition (4.13) of $\phi_{\mathbf{y}^*}$, we conclude from (4.15) that $\|\phi_{\mathbf{y}^*}\| = \|\phi_{\boldsymbol{\eta}^*}\|$ and that zero is a best approximation to $\phi_{\mathbf{y}^*}$ from $K(\mathbf{y}^*)$. Hence (4.15) becomes

$$\|\phi_{\mathbf{y}^*}\|^2 = \|\varphi_{\mathbf{y}^*}\|_{1,h} = \|\varphi_{\boldsymbol{\eta}^*}\|_{1,h},$$

which ocompletes the proof of the lemma, in view of Theorem 3.2 applied to the second problem in (4.14). ∎

Combining the results obtained in the lemmas we have

THEOREM 4.1. *There exists* $\mathbf{y}^* \in S^n$ *such that*

$$\|\phi_{\mathbf{y}^*}\| \leqslant \|\phi_{\mathbf{y}}\|, \qquad \mathbf{y} \in S^k, \quad k \leqslant n. \qquad (4.16)$$

A perfect spline ϕ_{y^*} *satisfies* (4.16) *if and only if* $\mathbf{y}^* \in S^n$ *and*

$$\|\phi_{\mathbf{y}^*}\| < \left\|\phi_{\mathbf{y}} - \sum_{i=1}^{k} a_i K(\cdot, y_i)\right\|, \qquad \mathbf{y} \in S^k, \quad \sum_{i=1}^{k} a_i^2 > 0, \quad k \leqslant n.$$

In particular, zero is the unique best approximation to ϕ_{y^*} *from* $K(\mathbf{y}^*)$.

Moreover, $\phi_{\mathbf{y}^*}$ *satisfies* (4.16) *if and only if* $\varphi_{\mathbf{y}^*}$ *is a perfect spline of minimum* L_h^1*-norm, and* $\|\phi_{\mathbf{y}^*}\|^2 = \|\varphi_{y^*}\|_{1,h}$.

For the case $h \equiv 1$ and $G(x, y) = G(x - y)$, uniqueness of the perfect spline of minimum L_h^1-norm is proved in [2]. From the equivalence between the problems (4.14), uniqueness of the perfect spline of minimum norm for norms of form (4.1) follows for a certain class of kernels $K(x, y)$ [2].

5. Applications to Approximation of Bivariate Functions and to n-Widths

The knots $y_1^*, \ldots, y_n^*$ and zeros $x_1^*, \ldots, x_n^*$ of the perfect spline of minimum norm play an important role in certain best approximation problems of TP kernels and in n-width problems related to these kernels. This is done in [12] for the L^p-norms, and in [11] for the sup-norm. Here we derive analogous results for monotone norms in the class $N(K)$, and for norms induced by inner products using similar ideas as in [12]. Two results play a central role in these derivations.

THE HOBBY–RICE THEOREM [3]. *Let* $u_1,\ldots,u_n \in L^1[a,b]$ *and let* $h \in L^\infty[a,b]$ *satisfy* (3.34). *Then there exists* $\mathbf{x} \in S^m[a,b]$, $m \leqslant n$, *such that*

$$\int_a^b u_i(x)\, h_{\mathbf{x}}(x)\, dx = 0, \qquad i = 1,\ldots, n. \tag{5.1}$$

THE BERNSTEIN COMPARISON THEOREM. *Let* $U = \{u_1,\ldots,u_n\} \subset C[a,b]$ *be a weak Chebyshev system, and let* $(f \pm g) \in C^+(U)$. *Then*

$$\inf_{u \in U} \|f - u\| \geqslant \inf_{u \in U} \|g - u\| \tag{5.2}$$

for any monotone norm $\|\cdot\|$.

This theorem was proved by Bernstein for polynomials and the sup-norm [1]. It was extended to monotone norms in [7] and to weak Chebyshev systems in [13].

As a by-product from the proof of this theorem we obtain ([7, 13])

RESULT D. *Let* $U \subset C^r[a,b]$ *and* $f, g \in C^r[a,b]$ *satisfy the condtions as above, and let* $\mathbf{x} = (a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b)$ *be of order* $r+1$ *such that*

$$\det \begin{pmatrix} u_1,\ldots, u_n \\ x_1,\ldots, x_n \end{pmatrix} > 0. \tag{5.3}$$

Then for any monotone norm $\|\cdot\|$

$$\|f - I_{\mathbf{x}} f\| \geqslant \|g - I_{\mathbf{x}} g\|, \tag{5.4}$$

where $I_{\mathbf{x}}$ *is the operator of interpolation at* $\mathbf{x}$ *by functions from* U.

For the special situation of perfect splines and monotone norms the last two results imply

COROLLARY 5.1. *Let* $K \in C^r([a,b] \times [c,d])$ *be* TP, *let* $h \in L^\infty[c,d]$ *satisfy* (3.1), *and let* $\phi_{\mathbf{y}}$ *for* $\mathbf{y} \in S^n[c,d]$ *be defined by* (3.4). *Then for any function of the form*

$$\phi = \int_c^d K(\cdot, y)\, \sigma(y)\, dy, \qquad |\sigma(y)| \leqslant h(y), \quad c \leqslant y \leqslant d, \quad \sigma \in L^\infty[c,d], \tag{5.5}$$

we have

$$\inf_{u \in K(\mathbf{y})} \|\phi_{\mathbf{y}} - u\| \geqslant \inf_{u \in K(\mathbf{y})} \|\phi - u\|, \tag{5.6}$$

$$\|\phi_{\mathbf{y}} - I_{\mathbf{x}} \phi_{\mathbf{y}}\| \geqslant \|\phi - I_{\mathbf{x}} \phi\|, \tag{5.7}$$

where $I_{\mathbf{x}}$ is the interpolation operator by funtions from $K(\mathbf{y})$ at a set of points $\mathbf{x} = (a \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant b)$ of order $r+1$ such that

$$\det K \begin{pmatrix} x_1, ..., x_n \\ y_1, ..., y_n \end{pmatrix} > 0. \tag{5.8}$$

Proof. It is enough to observe that $(-1)^n \phi_{\mathbf{y}} \pm \phi \in C^+(K(\mathbf{y}))$. This follows from Lemma 2.1 and the fact that $h_{\mathbf{y}}(y) \pm \sigma(y)$ alternates sign weakly at $y_1, ..., y_n$. ∎

For forms induced by inner products, the relation, revealed in Lemma 4.1, to approximation in the L_h^1-norm yields

LEMMA 5.1. *Under the assumptions of Section* 4,

$$\inf_{u \in K(\mathbf{y})} (\phi_{\mathbf{y}} - u, \phi_{\mathbf{y}} - u) \geqslant \inf_{u \in K(\mathbf{y})} (\phi - u, \phi - u), \tag{5.9}$$

for any $\mathbf{y} \in S^n$ *and any*

$$\phi = \int_a^b K(\cdot, y) \sigma(y)\, dy, \qquad |\sigma(y)| \leqslant h(y), \quad a \leqslant y \leqslant b, \quad \sigma \in L^\infty[a, b]. \tag{5.10}$$

Proof. By Lemma 4.1 if $u = \sum_{i=1}^n a_i K(\cdot, y_i)$ is the best approximation to $\phi_{\mathbf{y}}$ from $K(\mathbf{y})$, then

$$(\phi_{\mathbf{y}} - u, \phi_{\mathbf{y}} - u) = \| \varphi_{\mathbf{y}} - v \|_{1,h}, \qquad \varphi_{\mathbf{y}}(y_j) = v(y_j), \qquad j = 1, ..., n, \tag{5.11}$$

where $\varphi_{\mathbf{y}}(x) = \int_a^b G(x, y)\, h_{\mathbf{y}}(y)\, dy$, $v(x) = \sum_{i=1}^n a_i G(x, y_i)$, and $G(x, y) = G(y, x)$ is TP on $[a, b]^2$.

On the other hand, the best approximation $\hat{u} = \sum_{i=1}^n b_i K(\cdot, y_i)$ to ϕ from $K(\mathbf{y})$ satisfies the normal equations $(\phi - \hat{u}, K(\cdot, y_i)) = 0$, $i = 1, ..., n$ which are equivalent to

$$\varphi(y_i) - \hat{v}(y_i) = 0, \qquad i = 1, ..., n, \tag{5.12}$$

where $\varphi = \int_a^b G(\cdot, y)\, \sigma(y)\, dy$ and $\hat{v} = \sum_{i=1}^n b_i G(\cdot, y_i)$. Moreover,

$$(\phi - \hat{u}, \phi - \hat{u}) = (\phi - \hat{u}, \phi) = \int_a^b (\varphi - \hat{v}) \sigma\, dy \leqslant \| \varphi - \hat{v} \|_{1,h}. \tag{5.13}$$

Applying Corollary 5.1 to the TP kernel $G(x, y)$, the L_h^1-norm, and the perfect spline $\varphi_{\mathbf{y}}$, we conclude from (5.11), (5.12), (5.7), and (5.13) that

$$(\phi - \hat{u}, \phi - \hat{u}) \leqslant \| \varphi - \hat{v} \|_{1,h} \leqslant \| \varphi_{\mathbf{y}} - v \|_{1,h} = (\phi_{\mathbf{y}} - u, \phi_{\mathbf{y}} - u), \tag{5.14}$$

which completes the proof of the lemma. ∎

Another important result for certain n-width problems is

RESULT E [12]. *Let X be a space of functions defined on $[a,b]$, normed with a strictly convex norm $\|\cdot\|$. If $\phi_{\mathbf{y}} = \int_c^b K(\cdot, y)\, h_{\mathbf{y}}(y)\, dy \in X$ for all $\mathbf{y} \in S^n[c,d]$, then for any subspace of X of dimension n, X_n, there exists $\boldsymbol{\eta} \in S^n[c,d]$ such that*

$$\|\phi_{\boldsymbol{\eta}}\| = \inf_{u \in X_n} \|\phi_{\boldsymbol{\eta}} - u\|.$$

5.1. *Best Approximation of Bivariate* TP *Functions*

The first theorem deals with best approximation of bivariate functions in $C([a,b] \times [c,d])$ by tensor-product functions from $X \otimes L_h^1[c,d]$ in the tensor-product norm, where X is a linear space of functions defined on $[a,b]$, normed with a monotone norm.

THEOREM 5.1. *Let $K(x,y) \in C^{r,0}([a,b] \times [c,d])$, $h \in L^\infty[c,d]$, and a monotone norm $\|\cdot\|$ satisfy the assumptions of Theorem 3.1. Then a best approximation to $K(x,y)$ by functions of the form $\sum_{i=1}^n u_i(x)\, v_i(y) \in X \otimes L_h^1[c,d]$ in the norm*

$$|||f||| = \left\| \int_c^d |f(\cdot, y)|\, h(y)\, dy \right\| \tag{5.15}$$

is given by

$$\begin{aligned} &\sum_{i=1}^n \sum_{j=1}^n c_{ij} K(x_i^*, y) \cdot K(x, y_j^*) \\ &\quad \equiv K(x,y) - \left[\det K \begin{pmatrix} x, x_1^*, \dots, x_n^* \\ y, y_1^*, \dots, y_n^* \end{pmatrix} \Big/ \det K \begin{pmatrix} x_1^*, \dots, x_n^* \\ y_1^*, \dots, y_n^* \end{pmatrix} \right], \end{aligned} \tag{5.16}$$

where $c < y_1^* < \cdots < y_n^* < d$ and *$a \leqslant x_1^* \leqslant \cdots \leqslant x_n^* \leqslant b$ are the knots and zeros of a perfect spline of minimum norm $\|\cdot\|$.*

Proof. Given a function of the form $\sum_{i=1}^n u_i v_i \in X \otimes L_h^1[c,d]$, we obtain

$$\begin{aligned} &\int_c^d \left| K(x,y) - \sum_{i=1}^n u_i(x)\, v_i(y) \right| h(y)\, dy \\ &\quad \geqslant \left| \int_c^d \left[K(x,y) - \sum_{i=1}^n u_i(x)\, v_i(y) \right] h_{\boldsymbol{\eta}}(y)\, dy \right| \\ &\quad = |\phi_{\boldsymbol{\eta}}(x)|, \qquad x \in [a,b], \end{aligned}$$

where $h_{\boldsymbol{\eta}}$, $\boldsymbol{\eta} \in R^m[c,d]$, $m \leqslant n$, is an orthogonal function to $v_1, \dots, v_n$

guaranteed by the Hobby–Rice Theorem. Hence by the monotonicity of the norm $\|\cdot\|$ and by the definition of $\mathbf{y}^*$ and $\mathbf{x}^*$,

$$\left\|K(x,y) - \sum_{i=1}^{n} u_i(x)\, v_i(y)\right\|$$

$$\geqslant \|\phi_{\eta}\| \geqslant \|\phi_{\mathbf{y}^*}\| = \left\|\int_c^d K(\cdot, y)\, h_{\mathbf{y}^*}(y)\, dy\right\|$$

$$= \left\|\int_c^d \left[\det K \begin{pmatrix} x, x_1^*,..., x_n^* \\ y, y_1^*,..., y_n^* \end{pmatrix} \Big/ \det K \begin{pmatrix} x_1^*,..., x_n^* \\ y_1^*,..., y_n^* \end{pmatrix}\right] h_{\mathbf{y}^*}(y)\, dy\right\|$$

$$= \left\|K(x,y) - \sum_{i,j=1}^{n} c_{ij} K(x_i^*, y)\, K(x, y_j^*)\right\|. \quad \blacksquare$$

The rest of this section is devoted to various aspects of n-widths.

5.2. *On n-Widths Related to Monotone Norms*

We assume in this subsection that K, h, and the monotone norm $\|\cdot\|$ satisfy the assumptions of Theorem 3.1 and we consider the class of functions

$$K_h = \left\{\int_c^d K(x,y)\, \sigma(y)\, dy \;\middle|\; |\sigma(y)| \leqslant h(y), y \in [c,d], \sigma \in L^{\infty}[c,d]\right\}. \tag{5.17}$$

For this class we investigate two n-width problems; one related to the collection of functions

$$K\{(c,d)\} = \{K(\cdot, y) \mid y \in (c,d)\} \tag{5.18}$$

and the other related to all the functions in a space of functions X, $X \supset K\{(c,d)\}$, normed with the monotone norm $\|\cdot\|$.

The Kolmogorov n-width for this setup is

$$d_n(K_h, Y, \|\cdot\|) = \inf_{X_n \subset Y} \sup_{\phi \in K_h} \inf_{u \in X_n} \|\phi - u\|, \tag{5.19}$$

where Y is either $K\{(c,d)\}$ or X, and X_n is a space of dimension n spanned by n functions from Y. A space X_n attaining the infimum is called optimal. Under the assumptions of Theorem 3.1 we obtain only the weaker n-width $d_n(K_h, K\{(c,d)\}, \|\cdot\|)$. The result follows from property (3.22) of perfect splines of minimum norm, which is stronger than property (3.19) of minimality. If it is further assumed that the norm $\|\cdot\|$ is strictly convex, then $d_n(K_h, X, \|\cdot\|)$ can be obtained using only property (3.19) and Result E, as is done in [12] for the L^p-norms.

THEOREM 5.2. *Let K, h, and $\|\cdot\|$ satisfy the assumptions of Theorem* 3.1, *and let ϕ_{y^*}be a perfect spline of minimum norm. Then*

$$d_n(K_h, K\{(c,d)\}, \|\cdot\|) = \|\phi_{y^*}\| \tag{5.20}$$

and the space $K(\mathbf{y}^*)$ is optimal.

Proof. By Corollary 5.1

$$\sup_{\phi \in K_h} \inf_{u \in K(\mathbf{y})} \|\phi - u\| = \inf_{u \in K(\mathbf{y})} \|\phi_{\mathbf{y}} - u\|, \qquad \mathbf{y} \in S^n, \tag{5.21}$$

while by (3.22) of Theorem 3.2

$$\inf_{u \in K(\mathbf{y})} \|\phi_{\mathbf{y}} - u\| \geqslant \|\phi_{\mathbf{y}^*}\| = \inf_{u \in K(\mathbf{y}^*)} \|\phi_{\mathbf{y}^*} - u\|. \tag{5.22}$$

Combining (5.21) with (5.22), we obtain (5.20) and the optimality of $K(\mathbf{y}^*)$. ▮

THEOREM 5.3. *Let K, h, $\|\cdot\|$, and $\phi_{\mathbf{y}^*}$ be as in Theorem* 5.2. *If the norm $\|\cdot\|$ is strictly convex, then*

$$d_n(K_h, X, \|\cdot\|) = \|\phi_{\mathbf{y}^*}\| \tag{5.23}$$

and the space $K(\mathbf{y}^*)$ is optimal.

Proof. Given $X_n \subset X$, there esists by Result E, $\mathbf{y} \in S^m$, $m \leqslant n$, such that

$$\inf_{u \in X_n} \|\phi_{\mathbf{y}} - u\| = \|\phi_{\mathbf{y}}\|. \tag{5.24}$$

Hence

$$\sup_{\phi \in K_h} \inf_{u \in X_n} \|\phi - u\| \geqslant \|\phi_{\mathbf{y}}\|, \tag{5.25}$$

while by (3.19), (3.21), and Corollary 5.1

$$\sup_{\phi \in K_h} \inf_{u \in K(\mathbf{y}^*)} \|\phi - u\| \leqslant \inf_{u \in K(\mathbf{y}^*)} \|\phi_{\mathbf{y}^*} - u\| = \|\phi_{\mathbf{y}^*}\| \leqslant \|\phi_{\mathbf{y}}\|. \tag{5.26}$$

This completes the proof of the theorem. ▮

The conditions of the last theorem hold, in addition to the L_p-norms, $1 < p < \infty$, also for norms of form (2.17) with $1 < p_0, ..., p_r, q < \infty$ and for $K \in C^{1,0}([a,b] \times [c,d])$, TP, and nondegenerate of order 1 in x on $[a,b]$ and of order 0 in y on (c,d).

The Gelfand n-width for the class K_h, the norm $\|\cdot\|$, and a set of linear functionals F defined on $K\{(c, d)\}$ is defined as

$$d^n(K_h, F, \|\cdot\|) = \inf_{L_n \in F^n} \sup_{\phi \in K_h \cap L_n^\perp} \|\phi\|. \tag{5.27}$$

A space $L_n^\perp = \{f \mid l_i f = 0,\ l_i \in L_n,\ i = 1,..., n\}$ attaining the infimum is called optimal.

The Gelfand n-width is obtained under the more general situations of Theorem 5.2 without any further restrictions on the norm.

THEOREM 5.4. *Let K, h, and $\|\cdot\|$ satisfy the assumptions of Theorem 3.1, and let F be a set of linear functionals defined on $K\{(c, d)\}$ and containing $L_{\mathbf{x}^*} = \{l_{x_i^*},\ i = 1,..., n \mid l_{x_i^*} f = f^{(j)}(x_i),\ j = 1 - \operatorname{sgn}(x_i^* - x_{i-1}^*)\}$, with $a \leqslant x_1^* \leqslant \cdots \leqslant x_n^* \leqslant b$ the set of zeros of order 2 of a perfect spline of minimum norm $\phi_{\mathbf{y}^*}$. Then*

$$d^n(K_h, F, \|\cdot\|) = \|\phi_{\mathbf{y}^*}\|, \tag{5.28}$$

and the space $L_{\mathbf{x}^*}^\perp = \{f \mid l_{x_i^*} f = 0,\ i = 1,..., n\}$ is optimal.

Proof. For any $L_n \in F^n$, define $U_n = \{lK(\cdot, y) \mid l \in L_n\}$. Then by the Hobby–Rice theorem there exists $\mathbf{y} \in S^m$, $m \leqslant n$, such that

$$\int_c^d u(y)\, h_{\mathbf{y}}(y)\, dy = 0, \qquad u \in U_n, \tag{5.29}$$

or equivalently $\phi_{\mathbf{y}} \in L_n^\perp$. Hence

$$\sup_{\phi \in K_h \cap L_n^\perp} \|\phi\| \geqslant \|\phi_{\mathbf{y}}\| \geqslant \|\phi_{\mathbf{y}^*}\|, \tag{5.30}$$

while by (5.7) of Corollary 5.1

$$\|\phi_{\mathbf{y}^*}\| = \sup_{\phi \in K_h} \|\phi - I_{\mathbf{x}^*} \phi\| = \sup_{\phi \in K_h \cap L_{\mathbf{x}^*}^\perp} \|\phi\|, \tag{5.31}$$

where $I_{\mathbf{x}^*}$ is the interpolation operator from $K(\mathbf{y}^*)$ at the points $a \leqslant x_1^* \leqslant \cdots \leqslant x_n^* \leqslant b$. Combining (5.30) with (5.31) we obtain (5.28) and the optimality of $L_{\mathbf{x}^*}^\perp$. ∎

5.3. *On n-Widths Related to Norms Induced by Inner Products*

Similar results to the above can be obtained for norms induced by inner products, in view of Theorem 4.1 and Lemma 5.1. For the Kolmogorov n-width we have

THEOREM 5.5. *Let K, the inner product $(\ ,\ )$, and h satisfy the* *assumptions of Section* 4, *let* $\phi_{\mathbf{y}^*}$ *be minimal, and let* X *be an inner-product space with respect to* (,) *containing the set* $K\{(c,d)\}$. *Then for* $K_h = \{\phi \mid \phi = \int_a^b K(x,y)\,\sigma(y)\,dy, |\sigma(y)| \leqslant h(y), y \in [a,b], \sigma \in L^\infty[a,b]\}$ *we have*

$$d_n(K_h, X, (\ ,\ )^{1/2}) = (\phi_{\mathbf{y}^*}, \phi_{\mathbf{y}^*})^{1/2}, \tag{5.32}$$

and the space $K(\mathbf{y}^*)$ *is optimal.*

Proof. Since the norm $(f,f)^{1/2}$ is strictly convex, it follows from Result E that for $X_n \subset X$ there exists $\phi_{\mathbf{y}}$, $\mathbf{y} \in S^n[a,b]$, such that

$$\inf_{u \in X_n} (\phi_{\mathbf{y}} - u, \phi_{\mathbf{y}} - u) = (\phi_{\mathbf{y}}, \phi_{\mathbf{y}}). \tag{5.33}$$

On the other hand, by Theorem 4.1 and Lemma 5.1

$$\sup_{\phi \in K_h} \inf_{u \in K(\mathbf{y}^*)} (\phi - u, \phi - u) = (\phi_{\mathbf{y}^*}, \phi_{\mathbf{y}^*}) \leqslant (\phi_{\mathbf{y}}, \phi_{\mathbf{y}}), \tag{5.34}$$

which together with (5.33) completes the proof of the theorem. ∎

The Gelfand n-width for this setup is obtained similarly to that for monotone norms.

THEOREM 5.6. *Let* K, *the inner product* (,), h, X, *and* $\phi_{\mathbf{y}^*}$ *be as in Theorem* 5.5, *and let* $F \in X^*$ *be the set of all linear functionals in* X^* *mapping* $\{K(\cdot,y) \mid y \in [a,b]\}$ *into the reals. Then*

$$d^n(K_h, F, (\ ,\ )^{1/2}) = (\phi_{\mathbf{y}^*}, \phi_{\mathbf{y}^*})^{1/2} \tag{5.35}$$

and the space $K(\mathbf{y}^*)^\perp \equiv \{f \mid (f,g) = 0, g \in K(\mathbf{y}^*)\}$ *is optimal.*

Proof. For $L_n \in F^n$ the functions $U_n = \{lK(\cdot,y) \mid l \in L_n\}$ are real on $[a,b]$. Hence there exists by the Hobby–Rice theorem $\mathbf{y} \in S^m[a,b]$ such that $h_{\mathbf{y}}$ is orthogonal to U_n, or equivalently $\phi_{\mathbf{y}} \in L_n^\perp$. Applying Theorem 4.1, we conclude that

$$\sup_{\phi \in K_h \cap L_n^\perp} (\phi, \phi) \geqslant (\phi_{\mathbf{y}}, \phi_{\mathbf{y}}) \geqslant (\phi_{\mathbf{y}^*}, \phi_{\mathbf{y}^*}). \tag{5.36}$$

Now by assumption (iii) of Section 4, the set of linear functionals $\{l(f) = (f, K(\cdot,y)) \mid y \in [a,b]\}$ is in F, while for $\phi \in K_h \cap K(\mathbf{y}^*)^\perp$ it follows from Lemma 5.1 that

$$(\phi_{\mathbf{y}^*}, \phi_{\mathbf{y}^*}) = \sup_{\phi \in K_h \cap K(\mathbf{y}^*)^\perp} (\phi, \phi), \tag{5.37}$$

since the best approximation from $K(\mathbf{y}^*)$ to any $f \in K(\mathbf{y}^*)^\perp$ is zero. The claim of the theorem follows from (5.36) and (5.37). ∎

5.4. *On n-Widths Related to L_h^1-norms for Classes Defined by Monotone Norms*

More n-widths results can be obtained from the duality between perfect splines of minimum monotone norms and best approximations in the L^1-norm which is revealed in the proof of Lemma 3.1. These results deal with classes of the form

$$K_X = \left\{ \int_c^d K(x, y)\, g(y)\, dy \mid g \in X, \|g\| \leqslant 1 \right\}, \tag{5.38}$$

where $X \subset L^1[c, d]$ is a linear space normed with the monotone norm $\|\cdot\|$ and with n-widths of Kolmogorov and Gelfand type related to L_h^1-norms.

The class K_h in Subsection 5.2 can be viewed according to (5.38) as K_X with X the space $L^\infty[c, d]$ normed with the monotone norm $\|f\| \equiv \sup_{c \leqslant x \leqslant d} |f(x)/h(x)|$. Thus there is an overlap between the results there and those obtained hereafter.

In order to obtain these results, a dual norm to the monotone norm $\|\cdot\|$ has to be introduced.

LEMMA 5.2. *Let X be a linear space of functions defined on $[c, d]$, normed with a monotone norm $\|\cdot\|$. If $C[c, d] \subset X \subset L^1[c, d]$ and*

$$f \in X \Rightarrow |f| \operatorname{sgn}(g) \in X \qquad \textit{for any} \quad g \in C[c, d], \tag{5.39}$$

then the norm

$$|||g||| \equiv \sup_{\substack{f \in X \\ \|f\| = 1}} \left| \int_c^d f(y)\, g(y)\, dy \right| \tag{5.40}$$

is monotone on $C[c, d]$.

Proof. For $u, v \in C[c, d]$ such that $|u(y)| \leqslant |v(y)|$, $y \in [c, d]$, and for $f \in X$, $\|f\| = 1$,

$$\left| \int_c^d u(y) f(y)\, dy \right| \leqslant \int_c^d |u(y)|\, |f(y)|\, dy \leqslant \int_c^d |v(y)|\, |f(y)|\, dy$$
$$= \left| \int_c^d v(y)\, |f(y)| \operatorname{sgn}(v(y))\, dy \right|.$$

Now by (5.39), $|f| \operatorname{sgn}(v) \in X$ and by the monotonicity of the norm, $\| |f| \operatorname{sgn}(v) \| = \|f\| = 1$. Thus

$$\sup_{\substack{f\in X\\ \|f\|=1}} \left|\int_c^d f(y)\,u(y)\,dy\right| \leqslant \sup_{\substack{f\in X\\ \|f\|=1}} \left|\int_c^d f(y)\,v(y)\,dy\right|,$$

and the norm (5.40) is monotone. ▮

In the following we assume that X and the monotone norm in (5.38) satisfy the assumptions of Lemma 5.2. Furthermore, we assume that the dual norm $|||\cdot|||$ defined by (5.40) is in $N(K^{\mathrm{T}})$, where $K^{\mathrm{T}}(x,y)=K(y,x)$ is defined on $[c,d]\times[a,b]$ and satisfy the assumptions of Theorem 3.1. All these assumptions are satisfied by the L^p-norms and the corresponding space $L^p[c,d]$, $1\leqslant p\leqslant\infty$, $(p=\infty)$, and by kernels K which are TP on $[a,b]\times[c,d]$ and nondegenerate of order 0 in x on (a,b) and in y on (c,d) ($[c,d)$ or $(c,d]$). Also the norms of form (2.17),

$$\|f\|=\left[\sum_{i=0}^{m}\|f\|_{p_i}^{q}\right]^{1/q}, \qquad \|f\|_{p_i}=\|f\|_{L^{p_i}[\eta_i,\eta_{i+1}]}, \tag{5.41}$$

and their corresponding spaces

$$X=\{f \mid f|_{[\eta_i,\eta_{i+1}]}\in L^{p_i}[\eta_i,\eta_{i+1}],\, i=0,\ldots,m\}$$

with $1\leqslant p_0,\ldots,p_m<\infty$, $1\leqslant q\leqslant\infty$, and $c=\eta_0<\cdots<\eta_{m+1}=d$, satisfy the above assumptions, since the dual norms for these norms are [4]

$$|||g|||=\left[\sum_{i=0}^{m}\|f\|_{s_i}^{r}\right]^{1/r}, \qquad \|f\|_{s_i}=\|f\|_{L^{s_i}[\eta_i,\eta_{i+1}]} \tag{5.42}$$

with $r^{-1}+q^{-1}=1$ and $p_i^{-1}+s_i^{-1}=1$, $i=1,\ldots,m$.

The norms (5.42) are in $N(K^{\mathrm{T}})$ for kernels $K(x,y)$ in $C^{0,1}([a,b]\times[c,d])$ which are TP and nondegenerate of order 0 in x on (a,b) (or $[a,b]$) and of order 1 in y on $[c,d]$.

Under the above assumptions on K, x, and the norm $\|\cdot\|$, the Kolmogorov n-width is obtained for the space L_h^1.

THEOREM 5.7. *Let $h\in L^\infty[a,b]$ satisfy* (3.34), *let X and the norms $\|\cdot\|$ and $|||\cdot|||$ be as in Lemma* 5.2, *and let K^{T} and $|||\cdot|||$ satisfy the assumptions of Theorem* 3.1. *Then*

$$d_n(K_X, L_h^1, \|\cdot\|_{1,h})=|||\psi_{\xi^*}|||, \tag{5.43}$$

where $\psi_{\xi^*}=\int_a^b K(x,\cdot)\,h_{\xi^*}(x)\,dx$, $\boldsymbol{\xi}^*\in S^n[a,b]$, *is a perfect spline of minimum norm $|||\cdot|||$. An optimal space for* (5.45) *is the subspace $K(\boldsymbol{\eta}^*)$, where $c\leqslant\eta_1^*\leqslant\cdots\leqslant\eta_n^*\leqslant d$ are the zeros of ψ_{ξ^*}.*

Proof. For a given $X_n \subset L_h^1$, let $h_{\mathbf{x}}$, $\mathbf{x} \in S^m$, $m \leqslant n$, be an orthogonal function to X_n, guaranteed by the Hobby–Rice theorem. Then for $\phi \in K_X$

$$\inf_{u \in X_n} \|\phi - u\|_{1,h} \geqslant \left| \int_a^b \phi(x)\, h_{\mathbf{x}}(x)\, dx \right| = \left| \int_a^b \int_c^d K(x,y) f(y)\, h_{\mathbf{x}}(x)\, dx\, dy \right|$$

with $f \in X$, $\|f\| \leqslant 1$. Thus

$$\sup_{\phi \in K_X} \inf_{u \in X_n} \|\phi - u\|_{1,h} \geqslant \sup_{\substack{f \in X \\ \|f\| = 1}} \left| \int_c^d \psi_{\mathbf{x}}(y) f(y)\, dy \right| = |||\psi_{\mathbf{x}}||| \geqslant |||\psi_{\boldsymbol{\xi}^*}|||,$$

where $\psi_{\mathbf{x}} = \int_a^b K(x, \cdot)\ h_{\mathbf{x}}(x)\, dx$.

We complete the proof by demonstrating that

$$\sup_{\phi \in K_X} \inf_{u \in K(\boldsymbol{\eta}^*)} \|\phi - u\|_{1,h} \leqslant |||\psi_{\boldsymbol{\xi}^*}|||. \tag{5.44}$$

Let $\phi = \int_a^b K(\cdot, y) f(y)\, dy \in K_X$ and let

$$\Sigma \equiv \left\{ \sigma \;\middle|\; |\sigma(x)| = h(x),\, x \in (a,b),\, \int u\sigma\, dx = 0,\, u \in K(\boldsymbol{\eta}^*) \right\}.$$

By the dual characterization of best approximations,

$$\begin{aligned} \inf_{u \in K(\boldsymbol{\eta}^*)} \|\phi - u\|_{1,h} &= \sup_{\sigma \in \Sigma} \left| \int_a^b \phi(x)\, \sigma(x)\, dx \right| \\ &= \sup_{\sigma \in \Sigma} \left| \int_c^d \int_a^b [K(x,y)\, \sigma(x)\, dx] f(y)\, dy \right|, \end{aligned} \tag{5.45}$$

while by the definition of Σ, any function of the form $\psi = \int_a^b K(x, \cdot)\, \sigma(x)\, dx$ with $\sigma \in \Sigma$ vanishes at $c \leqslant \eta_1^* \leqslant \cdots \leqslant \eta_n^* \leqslant d$. Hence by (5.7) of Corollary 5.1 and definition (5.40) of the dual norm,

$$|||\psi_{\boldsymbol{\xi}^*}||| \geqslant \left|\left|\left| \int_a^b K(x,y)\, \sigma(x)\, dx \right|\right|\right| \geqslant \left| \int_c^d \int_a^b K(x,y)\, \sigma(x) f(y)\, dx\, dy \right|$$

for any $\sigma \in \Sigma$ and $f \in X$, $\|f\| = 1$. Therefore

$$\sup_{\sigma \in \Sigma} \left| \int_c^d \int_a^b K(x,y)\, \sigma(x) f(y)\, dx\, dy \right| \leqslant |||\psi_{\boldsymbol{\xi}^*}|||, \qquad f \in X, \quad \|f\| = 1,$$

which together with (5.45) proves (5.44). ∎

For the setup of the last theorem, we can obtain the Gelfand n-width not for $F = C^*[a, b]$ but only for the set of linear functionals

$$L\{(a, b)\} \equiv \{l \in C^*[a, b] \mid l(f) = f(x), x \in (a, b)\}. \tag{5.46}$$

This result is derived from the stronger property (3.22) of perfect splines of minimum norm as is the case for the Kolmogorov n-width of subsection 5.2. If we assume further that Result E holds for the dual norm $|||\cdot|||$, then the Gelfand n-width for $C^*[a, b]$ is obtained from the minimality of the perfect spline ψ_{ξ^*}.

THEOREM 5.8. *Let K, h, X, the norms $\|\cdot\|$ and $|||\cdot|||$, and ψ_{ξ^*} be as in Theorem 5.7. Then*

$$d^n(K_X, L\{(a, b)\}, \|\cdot\|_{1,h}) = |||\psi_{\xi^*}|||, \tag{5.47}$$

with an optimal space

$$L_{\xi^*}^{\perp} = \{f \mid f(\xi_i^*) = 0, i = 1,..., n\}. \tag{5.48}$$

Proof. For $\mathbf{x} \in S^n[a, b]$ let

$$L_{\mathbf{x}}^{\perp} = \{g \mid g(x_i) = 0,\ i = 1,..., n\}$$

and

$$M_{\mathbf{x}} = \left\{ f \in X \mid \|f\| \leqslant 1, \quad \int_c^d f(y)\, u(y)\, dy = 0, u \in K[\mathbf{x}] \right\}.$$

Now $\phi \in K_X \cap L_{\mathbf{x}}^{\perp}$ if and only if $\phi = \int_c^d K(\cdot, y) f(y)\, dy$ with $f \in M_{\mathbf{x}}$, and therefore

$$\begin{aligned} \sup_{\phi \in K_X \cap L_{\mathbf{x}}^{\perp}} \|\phi\|_{1,h} &= \sup_{f \in M_{\mathbf{x}}} \left\| \int_c^d K(x, y) f(y)\, dy \right\|_{1,h} \\ &\geqslant \sup_{f \in M_{\mathbf{x}}} \left| \int_a^b \left[\int_c^d K(x, y) f(y)\, dy \right] h_{\mathbf{x}}(x)\, dx \right| \\ &= \sup_{f \in M_{\mathbf{x}}} \left| \int_c^d \psi_{\mathbf{x}}(y) f(y)\, dy \right| = \inf_{u \in K[\mathbf{x}]} |||\psi_{\mathbf{x}} - u|||, \end{aligned}$$

where the last equality is the standard dual characterization of best approximations. Thus by (3.22) of Theorem 3.2

$$\sup_{\phi \in K_X \cap L_{\mathbf{x}}^{\perp}} \|\phi\|_{1,h} \geqslant \inf_{u \in K[\mathbf{x}]} |||\psi_{\mathbf{x}} - u||| \geqslant |||\psi_{\xi^*}|||. \tag{5.49}$$

To obtain the upper bound $|||\psi_{\xi^*}|||$ for the space $L_{\xi^*}^{\perp}$, observe that by Theorem 3.2 and Corollary 5.1

$$\inf_{u\in K[\xi^*]} |||\psi - u||| \leqslant |||\psi_{\xi^*}|||,$$

for any $\psi = \int_a^b K(x, \cdot)\,\sigma(x)\,dx$ with $|\sigma(x)| \leqslant h(x)$, $x \in [a, b]$. Hence for any $|\sigma(x)| = h(x)$, $x \in [a, b]$,

$$|||\psi_{\xi^*}||| \geqslant \sup_{f\in M_{\mathbf{x}^*}} \left| \int_a^b \left[\int_c^d K(x, y) f(y)\, dy \right] \sigma(x)\, dx \right|,$$

and therefore

$$|||\psi_{\xi^*}||| \geqslant \sup_{f\in M_{\xi^*}} \left\| \int_c^d K(x, y) f(y)\, dy \right\|_{1,h} = \sup_{\phi\in K_X\cap L_{\xi^*}^{\perp}} \|\phi\|_{1,h}. \quad \blacksquare$$

THEOREM 5.9. *Let K, h, X, the norms $\|\cdot\|$ and $|||\cdot|||$, and ψ_{ξ^*} be as in Theorem 5.7, and let the dual norm $|||\cdot|||$ be strictly convex. Then*

$$d^n(K_X, C^*[a, b], \|\cdot\|_{1,h}) = |||\psi_{\xi^*}||| \tag{5.50}$$

and the optimal space is $L_{\xi^*}^{\perp}$ (defined in (5.48)).

Proof. For $L_n \in (C^*[a, b])^n$ define $U_n = \{lK(\cdot, y) \mid l \in L_n\}$ and

$$M_n = \left\{ f \in X \,\middle|\, \|f\| \leqslant 1, \int_c^d f(y)\, u(y)\, dy = 0, u \in U_n \right\}. \tag{5.51}$$

By Result E there exists $\mathbf{x} \in S^n[a, b]$ such that

$$|||\psi_{\mathbf{x}}||| = \inf_{u\in U_n} |||\psi_{\mathbf{x}} - u|||. \tag{5.52}$$

Thus we obtain

$$\begin{aligned}
\sup_{\phi\in K_X\cap L_n^{\perp}} \|\phi\|_{1,h} &= \sup_{f\in M_n} \left\| \int_c^d K(x, y) f(y)\, dy \right\|_{1,h} \\
&\geqslant \sup_{f\in M_n} \left| \int_a^b \left[\int_c^d K(x, y) f(y)\, dy \right] h_{\mathbf{x}}(x)\, dx \right| \\
&= \sup_{f\in M_n} \left| \int_c^d \psi_{\mathbf{x}}(y) f(y)\, dy \right| = \inf_{u\in U_n} |||\psi_{\mathbf{x}} - u||| \\
&= |||\psi_{\mathbf{x}}||| \geqslant |||\psi_{\xi^*}|||,
\end{aligned}$$

where the last equalities are concluded from the dual characterization of best approximations and from (5.51)–(5.52). The optimality of $L_{\xi^*}^{\perp}$ follows from the last theorem. ∎

The norms (5.41) with $1 < q, p_0, ..., p_m < \infty$ and their dual norms (5.42) satisfy the assumptions of Theorem 5.9 for $K \in C^{0,1}([a, b] \times [c, d])$ which is TP and nondegenerate of order 0 in x on (a, b) and of order 1 in y on $[c, d]$.

Acknowledgement

The author thanks A. L. Brown for his valuable remarks which led to the present form of Theorem 3.1.

References

1. S. N. Bernstein, "Extremal Properties of the Polynomials of Best Approximation of a Continuous Function," Leningrad, (1937). [Russian]
2. D. Braess and N. Dyn, On the uniqueness of monosplines and perfect splines of least L_1- and L_2-norm, *J. Analyse Math.* **41** (1982), 217–233.
3. C. R. Hobby and R. Rice, A moment problem in L_1 approximation, *Proc. Amer. Math. Soc.* **16** (1965), 665–670.
4. A. Jakimovski and D. C. Russell, Representation of continuous linear functionals on a subspace of a countable Cartesian product of Banach spaces, *Studia Math.*, to appear.
5. S. Karlin, "Total Positivity," Vol. I, Stanford Univ. Press, Stanford, Calif., 1968.
6. S. Karlin, On a class of best nonlinear approximation problems and extended monosplines, in "Studies in Spline Functions and Approximation Theory," pp. 19–66, Academic Press, New York, 1976.
7. E. Kimchi and N. Richter-Dyn, Restricted range approximation of k-convex functions in monotone norms, *SIAM J. Numer. Anal.* **15** (1978), 1030–1038.
8. E. Kimchi and N. Richter-Dyn, A necessary condition for best approximation in monotone and sign-monotone norms, *J. Approx. Theory* **25** (1979), 169–175.
9. A. A. Melkman and C. A. Micchelli, Spline functions are optimal for L^2 n-width, *Illinois J. Math.* **22** (1978), 541–564.
10. C. A. Micchelli, Best L^1 approximation by weak Chebyshev systems and the uniqueness of interpolating perfect splines, *J. Approx. Theory* **19** (1977), 1–14.
11. C. A. Micchelli and A. Pinkus, On n-widths in L^∞, *Trans. Amer. Math. Soc.* **234** (1977), 139–174.
12. C. A. Micchelli and A. Pinkus, Some problems in the approximation of functions of two variables and n-widths of integral operators, *J. Approx. Theory* **24** (1978), 51–77.
13. A. Pinkus, "Bernstein Comparison Theorem and a Problem of Braess," *Aequationes Math.* **22** (1981), 318–320.
14. N. Richter-Dyn, Properties of minimal integration rules II, *SIAM J. Numer. Anal.* **8** (1971), 497–508.
15. N. Richter-Dyn, On best nonlinear approximation in sign-monotone norms and norms induced by inner products, *SIAM J. Numer. Anal.* **16** (1979), 612–622.